

REMARKS ON ELEMENTARY INTEGRAL CALCULUS FOR SUPERSMOOTH FUNCTIONS ON SUPERSPACE $\mathfrak{R}^{m|n}$

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Dedicated to the memory of late Professor Seizo ITO

ABSTRACT. After introducing Berezin integral for polynomials of odd variables, we develop the elementary integral calculus based on supersmooth functions on the superspace $\mathfrak{R}^{m|n}$. Here, \mathfrak{R} is the Fréchet-Grassmann algebra with countably infinite Grassmann generators, which plays the role of real number field \mathbb{R} . As is well-known that the formula of change of variables under integral sign is indispensable not only to treat PDE applying functional analytic method but also to introduce analysis on supermanifolds. But, if we define naively the integral for supersmooth functions, there exists discrepancy which should be ameliorated. Here, we extend the contour integral modifying the parameter space introduced basically by de Witt, Rogers and Vladimirov and Volovich.

1. INTRODUCTION: PROBLEM AND RESULTS

1.1. Problem. How to get the Feynman-like representation for the fundamental solution of the Dirac equation with external electro-magnetic potentials? (See, Inoue [9] for the free Dirac equation or Inoue [10] for the Weyl equation with external electro-magnetic field.) To answer this Feynman's question affirmatively but also to offer a prototype of new procedures to study other systems of PDE without diagonalizing coefficient matrices, we need to construct not only differential but also integral calculus based on a non-commutative algebra with countably many Grassmann generators.

Though, there are so many papers concerning elementary calculus prefixed “super-” which is based on Banach space, such as \mathfrak{B}_L or $\mathfrak{B}(= \mathfrak{B}_\infty)$, there is rather few dealing fully with elementary integral calculus based on Fréchet-Grassmann algebra, such as \mathfrak{R} introduced in Inoue and Maeda [12], Inoue [8, 11]. But a part of the elementary differential calculus based on \mathfrak{R} is mentioned, for example, in P.Bryant [2], Y.Choquet-Bruhat [3], S. Matsumoto and K. Kakazu [17], K. Yagi [26]. By the way, for the elementary differential calculus on a general Fréchet space, Hamilton's work [7] is transparent.

In our previous paper [11], we introduce and characterize the so-called supersmooth functions (alias superfield) on $\mathfrak{R}^{m|n}$. In order to treat certain systems of PDE without diagonalization, we regard matrices as differential operators acting on supersmooth functions and to apply method of functional analysis to that PDE, we need to develop integral calculus on $\mathfrak{R}^{m|n}$ which admit the formula of the change of variables under integral sign. Applying this integration theory, we may construct a parametrix having the representation of Fourier integral operator type with the phase function (roughly saying, with the matrix valued-phase function) satisfying the Hamilton-Jacobi equation. It should be remarked that since \mathfrak{R} is an infinite dimensional Fréchet space, we need a care.

Date: August 19, 2014.

1991 Mathematics Subject Classification. Primary 58C50, Secondary 58A50, 17A01.

Key words and phrases. Berezin integral, change of variables under integral sign.

Remark 1.1. *Matrices appeared are confined to $2^d \times 2^d$ -type, because we use Clifford algebras to expand matrices which have differential operator representations on Grassmann algebras. To treat matrices such as 3×3 , it seems necessary to develop another non-commutative space and analysis on it. See for example, Martin [16] and one may get some hints from Khare [13], or Campoamor-Stursberg, Rausch de Traubenberg [4].*

Concerning supersmooth functions only with even variables, we have the theory resembling to the integral in complex analysis, see, de Witt [6] and Rogers [18]. On the other hand, functions only with odd variables, we have the well-known Berezin integral.

In order to treat even and odd variables on equal footing, we need to mix naturally these variables, for example, the supersymmetric transformations are generated by mixing both variables. Therefore, we need to construct integration theory which admits the wide class of the change of variables under integral sign.

As is well-known, to study a scalar PDE by applying functional analysis, we use essentially the following tools: Taylor expansion, integration by parts, the formula for the change of variables under integral sign and Fourier transformation. Therefore, beside the elementary differential calculus, it is necessary to develop the elementary integral calculus on superspace $\mathfrak{R}^{m|n}$, both consist of the elementary superanalysis. But as is explained soon later, after defining differentials dx_j and $d\theta_k$ properly, we have the relations

$$\begin{cases} dx_j \wedge dx_k = -dx_k \wedge dx_j & \text{for even variables } \{x_j\}_{j=1}^m, \\ d\theta_j \wedge d\theta_k = d\theta_k \wedge d\theta_j & \text{for odd variables } \{\theta_k\}_{k=1}^n, \end{cases}$$

which differs from ordinary one.

Therefore, the integration containing odd variables doesn't necessarily follow from our conventional intuition.

Remark 1.2. *It is rather straight forward to extend the notion defined on Euclidian space to that on Banach space, but not so on Fréchet space. For example, the implicit function theorem is the typical one which is not extendable to general infinite dimensional Fréchet spaces without additional conditions. Therefore, we need a care to change the space from Roger's \mathfrak{B} to our \mathfrak{R} . Moreover, we have another algebraic operation, called the multiplication, in Fréchet-Grassmann or Banach-Grassmann algebras. Like the Fréchet differentiability on \mathbb{C} (=not only isomorphic to \mathbb{R}^2 but also have the multiplication) leads naturally to the notion of analyticity, the multiplication in those algebras yields the new notion called supersmoothness (see, [11]).*

Definition 1.1. *For a set $U \subset \mathbb{R}^m$, we define $\pi_B^{-1}(U) = \{X \in \mathfrak{R}^{m|0} \mid \pi_B(X) \in U\}$. A set $\mathfrak{U}_{\text{ev}} \subset \mathfrak{R}^{m|0}$ is called an “even superdomain” if $U = \pi_B(\mathfrak{U}_{\text{ev}}) \subset \mathbb{R}^m$ is open, connected and $\pi_B^{-1}(U) = \mathfrak{U}_{\text{ev}}$. U is denoted also by $\mathfrak{U}_{\text{ev}, B}$. When $\mathfrak{U} \subset \mathfrak{R}^{m|n}$ is represented by $\mathfrak{U} = \mathfrak{U}_{\text{ev}} \times \mathfrak{R}_{\text{od}}^n$ with an even superdomain $\mathfrak{U}_{\text{ev}} \subset \mathfrak{R}^{m|0}$, \mathfrak{U} is called a “superdomain” in $\mathfrak{R}^{m|n}$.*

Definition 1.2 (A naive definition of Berezin integral). *For a super domain $\mathfrak{U} = \mathfrak{U}_{\text{ev}} \times \mathfrak{R}^{0|n}$ and a supersmooth function $u(x, \theta) = \sum_{|a| \leq n} \theta^a u_a(x) : \mathfrak{U} \rightarrow \mathfrak{R}$, we “define” its integral as*

$$(1.1) \quad \mathfrak{B}\text{-}\iint_{\mathfrak{U}} dx d\theta u(x, \theta) = \int_{\mathfrak{U}_{\text{ev}}} dx \left(\int_{\mathfrak{R}^{0|n}} d\theta u(x, \theta) \right) = \int_{\pi_B(\mathfrak{U}_{\text{ev}})} dq u_{\bar{1}}(q),$$

where $\int_{\mathfrak{R}^{0|n}} d\theta u(x, \theta) = \frac{\partial}{\partial \theta_n} \cdots \frac{\partial}{\partial \theta_1} u(x, \theta) \Big|_{\theta_1 = \cdots = \theta_n = 0} = u_{\bar{1}}(x)$ and $\bar{1} = (\overbrace{1, \dots, 1}^n)$.

In the above, $u_{\bar{1}}(x)$ is the Grassmann continuation of $u_{\bar{1}}(q)$.

Desiring that the standard formula of the change of variables under integral sign (=CVF) holds by replacing standard Jacobian with super Jacobian(= super determinant of Jacobian matrix) on $\mathfrak{R}^{m|n}$, we have

Theorem 1.3. *Let $\mathfrak{U} = \mathfrak{U}_{\text{ev}} \times \mathfrak{R}^{0|n} \subset \mathfrak{R}_X^{m|n}$ and $\mathfrak{V} = \mathfrak{V}_{\text{ev}} \times \mathfrak{R}^{0|n} \subset \mathfrak{R}_Y^{m|n}$ be given. Let*

$$(1.2) \quad \varphi : \mathfrak{V} \ni Y = (y, \omega) \rightarrow X = (x, \theta) = (\varphi_0(y, \omega), \varphi_1(y, \omega)) \in \mathfrak{U}$$

be a supersmooth diffeomorphism from \mathfrak{V} onto \mathfrak{U} , that is,

$$(1.3) \quad \text{sdet} J(\varphi)(y, \omega) \neq 0 \quad \text{and} \quad \varphi(\mathfrak{V}) = \mathfrak{U} \quad \text{where} \quad J(\varphi)(y, \omega) = \begin{pmatrix} \frac{\partial \varphi_0(y, \omega)}{\partial y} & \frac{\partial \varphi_1(y, \omega)}{\partial y} \\ \frac{\partial \varphi_0(y, \omega)}{\partial \omega} & \frac{\partial \varphi_1(y, \omega)}{\partial \omega} \end{pmatrix}.$$

Then, for any function $u \in \mathcal{C}_{\text{SS}}(\mathfrak{U} : \mathfrak{C})$ with compact support, that is, $u(x, \theta) = \sum_{|a| \leq n} \theta^a u_a(x)$ where $u_a(x_B) \in C_0^\infty(\mathfrak{U}_{\text{ev}, B} : \mathfrak{R})$ for all $a \in \{0, 1\}^n$ except $a = \bar{1}$, we have CVF

$$(1.4) \quad \mathfrak{B} \! \! \! \int \! \! \! \int_{\mathfrak{U}} dx d\theta u(x, \theta) = \mathfrak{B} \! \! \! \int \! \! \! \int_{\varphi^{-1}(\mathfrak{U})} dy d\omega \text{sdet} J(\varphi)(y, \omega) u(\varphi(y, \omega)).$$

Remark 1.4. *Decomposing an even supermatrix M given by*

$$M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \quad \text{with} \quad \begin{cases} A = (a_{ij}), B = (b_{kl}), a_{ij}, b_{kl} \in \mathfrak{R}_{\text{ev}}, \\ C = (c_{i\ell}), D = (d_{kj}), c_{i\ell}, d_{kj} \in \mathfrak{R}_{\text{od}}, \end{cases}$$

we put

$$\text{sdet} M = \begin{cases} \det A \cdot \det(B - DA^{-1}C)^{-1} & \text{if } \det A_B \neq 0 \text{ where } A_B = (\pi_B a_{ij}), \\ \det(A - CB^{-1}D) \cdot \det B^{-1} & \text{if } \det B_B \neq 0 \text{ where } B_B = (\pi_B b_{kl}). \end{cases}$$

Remark 1.5. *Seemingly, this theorem implies that Berezin “measure” $D_0(x, \theta)$ is transformed by φ as*

$$(1.5) \quad (\varphi^* D_0(x, \theta))(y, \omega) = D_0(y, \omega) \cdot \text{sdet} J(\varphi)(y, \omega),$$

where

$$D_0(x, \theta) = dx_1 \wedge \cdots \wedge dx_m \otimes \frac{\partial}{\partial \theta_n} \cdots \frac{\partial}{\partial \theta_1} = dx_1 \cdots dx_m \cdot \partial_{\theta_n} \cdots \partial_{\theta_1} = dx \partial_{\theta}^{\bar{1}}, \quad D_0(y, \omega) = dy \partial_{\omega}^{\bar{1}}.$$

But this assertion is shown to be false in general by the following example. Moreover, we remark also that the condition of “the compact supportness of integrands” above, seems not only cumbersome from conventional point of view but also fatal in holomorphic category.

Example 1.1. *Let $\mathfrak{U} = \pi_B^{-1}(\Omega) \times \mathfrak{R}_{\text{od}}^2 \subset \mathfrak{R}^{1|2}$ with $\Omega = (0, 1)$, $\pi_B : \mathfrak{R}^{1|0} \rightarrow \mathbb{R}$ and let u be supersmooth on $\mathfrak{R}^{1|2}$ with value in \mathfrak{R} such that $u(x, \theta) = u_0(x) + \theta_1 \theta_2 u_1(x)$. Then, we have*

$$\mathfrak{B} \! \! \! \int \! \! \! \int_{\mathfrak{U}} dx d\theta u(x, \theta) = \int_{\Omega} dx \int d\theta u(x, \theta) = \int_{\pi_B^{-1}(\Omega)} dx u_1(x) = \int_{\Omega} dq u_1(q).$$

But, if we use the coordinate change

$$(1.6) \quad \varphi : (y, \omega) \rightarrow (x, \theta) \quad \text{with} \quad x = y + \omega_1 \omega_2 \phi(y), \quad \theta_k = \omega_k : \mathfrak{U} \rightarrow \mathfrak{U}$$

whose Berezinian is

$$\text{Ber}(\varphi)(y, \omega) = \text{sdet} J(\varphi)(y, \omega) = 1 + \omega_1 \omega_2 \phi'(y) \quad \text{where} \quad J(\varphi)(y, \omega) = \begin{pmatrix} 1 + \omega_1 \omega_2 \phi'(y) & 0 & 0 \\ \omega_2 \phi(y) & 1 & 0 \\ -\omega_1 \phi(y) & 0 & 1 \end{pmatrix},$$

and if we assume that the formula (1.4) holds, then since

$$u(\varphi(y, \omega)) = u_0(y + \omega_1 \omega_2 \phi(y)) + \omega_1 \omega_2 u_1(y + \omega_1 \omega_2 \phi(y)) = u_0(y) + \omega_1 \omega_2 (\phi(y) u_0'(y) + u_1(y)),$$

$$\text{and} \quad (1 + \omega_1 \omega_2 \phi'(y)) u(\varphi(y, \omega)) = u_0(y) + \omega_1 \omega_2 (\phi(y) u_0'(y) + \phi'(y) u_0(y) + u_1(y)),$$

we have

$$\mathfrak{B}\text{-}\iint_{\varphi^{-1}(\mathfrak{U})} dy d\omega (1 + \omega_1 \omega_2 \phi'(y)) u(\varphi(y, \omega)) = \int_{\pi_B^{-1}(\Omega)} dy (\phi(y) u_0(y))' + \int_{\pi_B^{-1}(\Omega)} dx u_1(x).$$

Therefore, if $\int_{\pi_B^{-1}(\Omega)} dy (\phi(y) u_0(y))' \neq 0$, then $\iint_{\mathfrak{U}} D_0(x, \theta) u(x, \theta) \neq \iint_{\varphi^{-1}(\mathfrak{U})} D_0(y, \omega) u(\varphi(y, \omega))$. This implies that if we apply (1.1) as definition, the change of variables formula doesn't hold when, for example, the integrand hasn't compact support.

1.2. Results. Though there seems several methods (for example, due to Rothstein [24], Zirnbauer [27]) to remedy such inconsistency in case $\mathfrak{B}_L^{m|n}$ or $\mathfrak{B}^{m|n}$, we modify Vladimirov and Volovich [25] to get our results.

Definition 1.3 (Parameter set, paths and integral). *We prepare a domain Ω in \mathbb{R}^m and put $\tilde{\Omega} = \Omega \times \mathfrak{R}_{\text{od}}^n$, called a parameter set.*

(1) Let $\gamma \in C^\infty(\tilde{\Omega} : \mathfrak{R}^{m|n})$ with $\gamma(q, \vartheta) = (\gamma_0(q, \vartheta), \gamma_1(q, \vartheta)) = (\gamma_{0,j}(q, \vartheta), \gamma_{1,k}(q, \vartheta))_{\substack{j=1, \dots, m \\ k=1, \dots, n}}$ be given such that

$$\gamma_{0,j}(q, \vartheta) = \sum_{|a| \leq n} \vartheta^a \gamma_{0,j,a}(q) \in \mathfrak{R}_{\text{ev}}, \quad \gamma_{1,k}(q, \vartheta) = \sum_{|a| \leq n} \vartheta^a \gamma_{1,k,a}(q) \in \mathfrak{R}_{\text{od}}$$

where

$$\gamma_{0,j,a}(q) = \sum_{|\mathbf{I}|=|a| \pmod{2}} \gamma_{0,j,a,\mathbf{I}}(q) \sigma^{\mathbf{I}}, \quad \gamma_{1,k,a}(q) = \sum_{|\mathbf{J}|=|a|+1 \pmod{2}} \gamma_{1,k,a,\mathbf{J}}(q) \sigma^{\mathbf{J}}$$

with

$$\gamma_{0,a,\mathbf{I}}(q), \gamma_{1,a,\mathbf{J}}(q) \in C^\infty(\Omega : \mathbb{C}^n) \quad \text{and} \quad \gamma_{0,\bar{0},\bar{0}}(q) \in C^\infty(\Omega : \mathbb{R}^m), \quad \bar{0} = (0, \dots, 0), \quad \tilde{0} = (0, \dots).$$

In case

$$\text{sdet } J(\gamma)(q, \vartheta) \neq 0 \quad \text{where} \quad J(\gamma)(q, \vartheta) = \frac{\partial \gamma(q, \vartheta)}{\partial(q, \vartheta)} = \begin{pmatrix} \frac{\partial \gamma_0(q, \vartheta)}{\partial q} & \frac{\partial \gamma_1(q, \vartheta)}{\partial q} \\ \frac{\partial \gamma_0(q, \vartheta)}{\partial \vartheta} & \frac{\partial \gamma_1(q, \vartheta)}{\partial \vartheta} \end{pmatrix},$$

we call this γ as a “path” from $\tilde{\Omega}$ into $\mathfrak{R}^{m|n}$ and its image is called a foliated singular manifold:

$$\mathfrak{M} = \mathfrak{M}(\gamma, \Omega) = \gamma(\tilde{\Omega}) = \{(x, \theta) \in \mathfrak{R}^{m|n} \mid x = \gamma_0(q, \vartheta), \theta = \gamma_1(q, \vartheta), q \in \Omega, \vartheta \in \mathfrak{R}_{\text{od}}^n\}.$$

(2) For a supersmooth function $u(x, \theta) = \sum_{|a| \leq n} \theta^a u_a(x)$ defined on \mathfrak{M} , we call the following expression as “the integral of the function $u(x, \theta)$ over the foliated singular manifold \mathfrak{M} ”;

$$(1.7) \quad \mathfrak{M}\text{-}\iint_{\mathfrak{M}} dx d\theta u(x, \theta) = \int_{\mathfrak{R}_{\text{od}}^n} d\vartheta \left[\int_{\Omega} dq \text{sdet } J(\gamma)(q, \vartheta) u(\gamma(q, \vartheta)) \right].$$

Here, we assume that for each $\eta \in \mathfrak{R}_{\text{od}}^n$, the integral in the bracket $[\dots]$ above exists as the integral on Ω .

Definition 1.4. Let two foliated singular manifolds $\mathfrak{M} = \gamma(\tilde{\Omega})$ and $\mathfrak{M}_1 = \gamma_1(\tilde{\Omega}_1)$ be given. We call these are superdiffeomorphic if there exist a diffeomorphism $\phi : \tilde{\Omega}_1 \rightarrow \tilde{\Omega}$ and $\varphi : \mathfrak{M}_1 \rightarrow \mathfrak{M}$ such that $\gamma_1 = \varphi^{-1} \circ \gamma \circ \phi$.

$$\begin{array}{ccc} \tilde{\Omega} & \xrightarrow{\gamma} & \mathfrak{M} = \gamma(\tilde{\Omega}) \\ \phi \uparrow & & \uparrow \varphi \\ \tilde{\Omega}_1 & \xrightarrow{\gamma_1} & \mathfrak{M}_1 = \gamma_1(\tilde{\Omega}_1). \end{array}$$

Theorem 1.6. Let

$$(1.8) \quad \varphi : (y, \omega) \rightarrow (x, \theta) \quad \text{with} \quad x = \varphi_0(y, \omega), \quad \theta = \varphi_1(y, \omega)$$

be a supersmooth diffeomorphism from the neighbourhood \mathfrak{D}_1 of the foliated singular manifold $\mathfrak{N}(\delta, \Omega)$ in $\mathfrak{R}^{m|n}$ onto the neighbourhood \mathfrak{D} of the foliated singular manifold $\mathfrak{M}(\gamma, \Omega)$ in $\mathfrak{R}^{m|n}$, that is, $\mathfrak{M} = \varphi(\mathfrak{N})$ and $\text{sdet } J(\varphi) \neq 0$. We assume moreover that $\delta = \varphi^{-1} \circ \gamma$ with $\text{sdet } J(\gamma) \neq 0$.

Then, for any function $u \in \mathcal{C}_{\text{SS}}(\mathfrak{D} : \mathfrak{R})$ which is integrable on \mathfrak{M} , we have CVF

$$(1.9) \quad \mathfrak{W}\!\!\!\int\!\!\!\int_{\mathfrak{M}} dx d\theta u(x, \theta) = \mathfrak{W}\!\!\!\int\!\!\!\int_{\varphi^{-1}(\mathfrak{M})} dy d\omega \text{sdet } J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)).$$

Remark 1.7. An analogous statement on Banach-Grassmann algebra $\mathfrak{B}_L^{m|n}$ is proved in [25] under the condition that the set $\{x \in \mathfrak{B}_{L0}^m \mid x = \gamma(q, \vartheta), q \in \Omega\}$ is independent for each $\vartheta \in \mathfrak{B}_{L1}^n$.

Remark 1.8. The formulas (1.4) and (1.9) look same but their underlying definitions (1.1) and (1.7) are different! This is related to the problem “How to consider the body of supermanifolds?” (see, Catenacci, Reina and Teofilatto [5]).

2. ILLUSTRATION: RESOLUTION OF INCONSISTENCY OF EXAMPLE 1.1 BY CONTOUR INTEGRAL

From Theorem 1.6, we have the following interpretation:

Let $\tilde{\Omega} = \Omega \times \mathfrak{R}_{\text{od}}^2$ with $\Omega = (0, 1)$ be given. Defining $\gamma : \tilde{\Omega} \rightarrow \mathfrak{M}$ by

$$\gamma : \tilde{\Omega} \ni (q, \vartheta) \rightarrow (x, \theta) = (\gamma_{\bar{0}}(q, \vartheta), \gamma_{\bar{1}}(q, \vartheta)) = \gamma(q, \vartheta),$$

we may consider $\mathfrak{M} = \{(x, \theta) \in \mathfrak{R}^{1|2} \mid \pi_{\text{B}}(x) \in \Omega, \theta \in \mathfrak{R}_{\text{od}}^2\}$ as a singular foliated manifold $\gamma(\tilde{\Omega})$ in $\mathfrak{R}^{1|2}$. Prepare another singular foliated manifold $\mathfrak{N} = \delta(\tilde{\Omega})$ in $\mathfrak{R}^{1|2}$ with a superdiffeomorphism

$$\varphi : \delta(\tilde{\Omega}) \ni (y, \omega) \rightarrow \varphi(y, \omega) = (x, \theta) \in \gamma(\tilde{\Omega}),$$

given by

$$\begin{cases} x = \varphi_{\bar{0}}(y, \omega) = y + \omega_1 \omega_2 \phi(y), \\ \theta_1 = \varphi_{\bar{1},1}(y, \omega) = \omega_1, \theta_2 = \varphi_{\bar{1},2}(y, \omega) = \omega_2, \end{cases}$$

and

$$\delta = \varphi^{-1} \circ \gamma : (q, \vartheta) \rightarrow (q - \vartheta_1 \vartheta_2 \phi(q), \vartheta) = (\delta_{\bar{0}}(q, \vartheta), \delta_{\bar{1}}(q, \vartheta)) = (y, \omega).$$

Then, we have $\mathfrak{N} = \varphi^{-1}(\mathfrak{M})$ and

$$J(\varphi)(y, \omega) = \begin{pmatrix} 1 + \omega_1 \omega_2 \phi'(y) & 0 & 0 \\ \omega_2 \phi(y) & 1 & 0 \\ -\omega_1 \phi(y) & 0 & 1 \end{pmatrix}, \quad J(\gamma)(q, \vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J(\delta)(q, \vartheta) = \begin{pmatrix} 1 - \vartheta_1 \vartheta_2 \phi'(q) & 0 & 0 \\ -\vartheta_2 \phi(q) & 1 & 0 \\ \vartheta_1 \phi(q) & 0 & 1 \end{pmatrix}.$$

In this case, for $u(x, \theta) = u_{\bar{0}}(x) + \theta_1 \theta_2 u_{\bar{1}}(x)$, we have

$$\begin{aligned} \mathfrak{W}\!\!\!\int\!\!\!\int_{\mathfrak{M}} dx d\theta u(x, \theta) &= \int_{\mathfrak{R}_{\text{od}}^2} d\vartheta \left[\int_{\Omega} dq \text{sdet } J(\gamma)(q, \vartheta) u(\gamma(q, \vartheta)) \right] \\ &= \int_0^1 dq \int_{\mathfrak{R}_{\text{od}}^2} d\vartheta u(q, \vartheta) = \int_0^1 dq \frac{\partial}{\partial \vartheta_2} \frac{\partial}{\partial \vartheta_1} u(q, \vartheta) \Big|_{\vartheta=0} = \int_0^1 dq u_{\bar{1}}(q), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{W}\!\!\!\int\!\!\!\int_{\mathfrak{N}} dy d\omega (\varphi^* u)(y, \omega) &= \int_{\tilde{\Omega}} dq d\vartheta \text{sdet } J(\delta)(q, \vartheta) [\text{sdet } J(\varphi)(y, \omega) u(\varphi(y, \omega))]_{(y, \omega) = \delta(q, \vartheta)} \\ (2.1) \quad &= \int_0^1 dq \left[\int_{\mathfrak{R}_{\text{od}}^2} d\vartheta \text{sdet } J(\delta)(q, \vartheta) [\text{sdet } J(\varphi)(y, \omega) u(\varphi(y, \omega))]_{(y, \omega) = \delta(q, \vartheta)} \right] \\ &= \int_0^1 dq \int_{\mathfrak{R}_{\text{od}}^2} d\vartheta u(q, \vartheta). \end{aligned}$$

Therefore, we have the following result with no condition on the support of u :

$$\begin{aligned}
\mathfrak{W}\!\!\!\int\!\!\int_{\mathfrak{M}} dx d\theta u(x, \theta) &= \int\!\!\int_{\tilde{\Omega}} dq d\vartheta \operatorname{sdet} J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)) \\
&= \int\!\!\int_{\tilde{\Omega}} dq d\vartheta \operatorname{sdet} J(\delta)(q, \vartheta) \left[\operatorname{sdet} J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)) \right] \Big|_{(y, \omega) = \delta(q, \vartheta)} \\
&= \mathfrak{W}\!\!\!\int\!\!\int_{\varphi^{-1}(\mathfrak{M})} dy d\omega \operatorname{sdet} J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)) = \mathfrak{W}\!\!\!\int\!\!\int_{\mathfrak{M}} dy d\omega (\varphi^* u)(y, \omega). \quad \square
\end{aligned}$$

Remark 2.1. For the future use, we calculate more precisely:

$$\begin{aligned}
\operatorname{sdet} J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)) &= (1 + \omega_1 \omega_2 \phi'(y)) [u_{\bar{0}}(y + \omega_1 \omega_2 \phi(y)) + \omega_1 \omega_2 u_{\bar{1}}(y + \omega_1 \omega_2 \phi(y))] \\
(2.2) \quad &= (1 + \omega_1 \omega_2 \phi'(y)) [u_{\bar{0}}(y) + \omega_1 \omega_2 (\phi(y) u'_{\bar{0}}(y) + u_{\bar{1}}(y))] \\
&= u_{\bar{0}}(y) + \omega_1 \omega_2 [\underbrace{(\phi(y) u_{\bar{0}}(y))'}_{\text{term}} + u_{\bar{1}}(y)],
\end{aligned}$$

and putting $(y, \omega) = \delta(q, \vartheta)$, we have

$$\begin{aligned}
\operatorname{sdet} J(\delta)(q, \vartheta) \left[\operatorname{sdet} J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)) \right]_{(y, \omega) = \delta(q, \vartheta)} &= (1 - \vartheta_1 \vartheta_2 \phi'(q)) (u_{\bar{0}}(y) + \omega_1 \omega_2 [(\phi(y) u_{\bar{0}}(y))' + u_{\bar{1}}(y)]) \Big|_{\substack{y=q-\vartheta_1 \vartheta_2 \phi(q), \\ \omega_1=\vartheta_1, \omega_2=\vartheta_2}} \\
(2.3) \quad &= \underbrace{(1 - \vartheta_1 \vartheta_2 \phi'(q))}_{\text{term}} [u_{\bar{0}}(q) - \vartheta_1 \vartheta_2 \phi(q) u'_{\bar{0}}(q) + \vartheta_1 \vartheta_2 [(\phi(q) u_{\bar{0}}(q))' + u_{\bar{1}}(q)]] \\
&= u_{\bar{0}}(q) + \vartheta_1 \vartheta_2 [(\phi(q) u_{\bar{0}}(q))' + u_{\bar{1}}(q) - \underbrace{(\phi(q) u_{\bar{0}}(q))'}_{\text{term}}] = u_{\bar{0}}(q) + \vartheta_1 \vartheta_2 u_{\bar{1}}(q).
\end{aligned}$$

Or since $u(\varphi(y, \omega))_{(y, \omega) = \delta(q, \vartheta)} = u(q, \vartheta)$ and

$$\operatorname{sdet} J(\delta)(q, \vartheta) \cdot \operatorname{sdet} J(\varphi)(\delta(q, \vartheta)) = (1 - \vartheta_1 \vartheta_2 \phi'(q)) (1 + \vartheta_1 \vartheta_2 \phi'(q)) = 1,$$

we have the result.

Therefore, the appearance of the term $\omega_1 \omega_2 (\phi(y) u_{\bar{0}}(y))'$ in (2.2) is the very reason of inconsistency.

3. INTEGRATION W.R.T. EVEN VARIABLES

3.1. One dimensional case as a prototype. We recall the idea of the contour integral noted in Rogers [21].

Contour integrals are a means of “pulling back” an integral in a space that is algebraically (as well as possibly geometrically) more complicated than \mathbb{R}^m . A familiar example, of course, is complex contour integration; if $\gamma : [0, 1] \rightarrow \mathbb{C}$ is piecewise C^1 and $f : \mathbb{C} \rightarrow \mathbb{C}$, one has the one-dimensional contour integral

$$\int_{\gamma} f(z) dz = \int_0^1 f(\gamma(t)) \cdot \gamma'(t) dt = \int_0^1 dt \gamma'(t) \cdot f(\gamma(t)).$$

This involves the algebraic structure of \mathbb{C} because the right-hand side of above includes multiplication \cdot of complex numbers.

We follow this idea to define the integral of a supersmooth function $u(x)$ on an even superdomain $\mathfrak{U}_{\text{ev}} \subset \mathfrak{R}^{m|0} = \mathfrak{R}_{\text{ev}}^m$ (see also, Rogers [19, 20, 22] and Vladimirov and Volovich [25]).

Definition 3.1. Let $u(x)$ be a supersmooth function defined on an even superdomain $\mathfrak{U}_{\text{ev}} \subset \mathfrak{R}^{1|0}$ such that $[a, b] \subset \pi_{\text{B}}(\mathfrak{U}_{\text{ev}})$. Let $\lambda = \lambda_{\text{B}} + \lambda_{\text{S}}, \mu = \mu_{\text{B}} + \mu_{\text{S}} \in \mathfrak{U}_{\text{ev}}$ with $\lambda_{\text{B}} = a, \mu_{\text{B}} = b$, and let a continuous and piecewise C^1 -curve $\gamma : [a, b] \rightarrow \mathfrak{U}_{\text{ev}}$ be given such that $\gamma(a) = \lambda, \gamma(b) = \mu$. We define

$$(3.1) \quad \int_{\gamma} dx u(x) = \int_a^b dt \gamma'(t) \cdot u(\gamma(t)) \in \mathfrak{C} \quad \text{with} \quad \gamma'(t) = \dot{\gamma}(t) = \frac{d\gamma(t)}{dt}$$

and call it the integral of u along the curve γ .

Using the integration by parts for functions on \mathbb{R} , we get the following fundamental result.

Proposition 3.1 (p.7 of de Witt [6]). *Let $u(t) \in C^\infty([a, b] : \mathfrak{C})$ and $U(t) \in C^\infty([a, b] : \mathfrak{C})$ be given such that $U'(t) = u(t)$ on $[a, b]$. We denote the Grassmann continuations of them as $\tilde{u}(x)$ and $\tilde{U}(x)$. Then, for any continuous and piecewise C^1 -curve $\gamma : [a, b] \rightarrow \mathfrak{U}_{\text{ev}} \subset \mathfrak{R}^{1|0}$ such that $[a, b] \subset \pi_B(\mathfrak{U}_{\text{ev}})$ and $\gamma(a) = \lambda$, $\gamma(b) = \mu$ with $\lambda_B = a$, $\mu_B = b$, we have*

$$(3.2) \quad \int_{\gamma} dx \tilde{u}(x) = \tilde{U}(\lambda) - \tilde{U}(\mu).$$

Proof. Denoting $\dot{\gamma}_B(t) = \frac{d}{dt}\gamma_B(t)$, etc. and by definition, we get

$$\begin{aligned} \int_a^b dt \gamma'(t) u(\gamma(t)) &= \int_a^b dt (\dot{\gamma}_B(t) + \dot{\gamma}_S(t)) \sum_{\ell \geq 0} \frac{1}{\ell!} u^{(\ell)}(\gamma_B(t)) \gamma_S(t)^\ell \\ &= \int_a^b dt \dot{\gamma}_B(t) u(\gamma_B(t)) + \int_a^b dt \dot{\gamma}_B(t) \sum_{k \geq 1} \frac{1}{k!} u^{(k)}(\gamma_B(t)) \gamma_S(t)^k \\ &\quad + \int_a^b dt \sum_{\ell \geq 0} \frac{1}{\ell!} u^{(\ell)}(\gamma_B(t)) \dot{\gamma}_S(t) \gamma_S(t)^\ell \\ &= U(b) - U(a) + \sum_{\ell \geq 0} \frac{1}{(\ell+1)!} \left\{ U^{(\ell+1)}(b) \mu_S^{\ell+1} - U^{(\ell+1)}(a) \lambda_S^{\ell+1} \right\} \\ &= \tilde{U}(\mu) - \tilde{U}(\lambda). \end{aligned}$$

Here, we used the integration by parts formula for functions on \mathbb{R} valued in Fréchet space [7]:

$$\begin{aligned} \int_a^b dt u^{(\ell)}(\gamma_B(t)) \dot{\gamma}_S(t) \gamma_S(t)^\ell &= \int_a^b dt u^{(\ell)}(\gamma_B(t)) \frac{d}{dt} \frac{\gamma_S(t)^{\ell+1}}{\ell+1} \\ &= - \int_a^b dt \dot{\gamma}_B(t) u^{(\ell+1)}(\gamma_B(t)) \frac{\gamma_S(t)^{\ell+1}}{\ell+1} + U^{(\ell+1)}(b) \frac{\mu_S^{\ell+1}}{\ell+1} - U^{(\ell+1)}(a) \frac{\lambda_S^{\ell+1}}{\ell+1}. \quad \square \end{aligned}$$

Remark 3.2. Unless there occurs the confusion, we denote simply $\tilde{u}(x)$, $\tilde{U}(x)$ as $u(x)$, $U(x)$, respectively.

Lemma 3.3 (Lemma 3.9 in [19] on \mathfrak{B}_L). (a) (reparametrization of paths) Let $\gamma : [a, b] \rightarrow \mathfrak{R}_{\text{ev}}$ be a path in \mathfrak{R}_{ev} and let $c, d \in \mathbb{R}$. Also let $\phi : [c, d] \rightarrow [a, b]$ be C^1 with $\phi(c) = a$, $\phi(d) = b$ and $\phi'(s) > 0$ for all $s \in [c, d]$. Then

$$\int_{\gamma} dx u(x) = \int_{\gamma \circ \phi} dx u(x).$$

(b)(sum of paths) Let $\gamma_1 : [a, b] \rightarrow \mathfrak{R}_{\text{ev}}$ and $\gamma_2 : [c, d] \rightarrow \mathfrak{R}_{\text{ev}}$ be two paths with $\gamma_1(b) = \gamma_2(c)$. Also define $\gamma_1 + \gamma_2$ to be the path $\gamma_1 + \gamma_2 : [a, b + d - c] \rightarrow \mathfrak{R}_{\text{ev}}$ defined by

$$\gamma_1 + \gamma_2(t) = \begin{cases} \gamma_1(t), & a \leq t \leq b, \\ \gamma_2(t - b + c), & b \leq t \leq b + d - c. \end{cases}$$

Then if \mathfrak{U}_{ev} is open in \mathfrak{R}_{ev} , $u : \mathfrak{U}_{\text{ev}} \rightarrow \mathfrak{R}$ is in \mathcal{C}_{SS} and $\gamma_1([a, b]) \subset \mathfrak{U}_{\text{ev}}$, $\gamma_2([c, d]) \subset \mathfrak{U}_{\text{ev}}$,

$$\int_{\gamma_1 + \gamma_2} dx u(x) = \int_{\gamma_1} dx u(x) + \int_{\gamma_2} dx u(x).$$

(c)(inverse of a path) Let $\gamma : [a, b] \rightarrow \mathfrak{R}_{\text{ev}}$ be a path in \mathfrak{R}_{ev} . Define the curve $-\gamma : [a, b] \rightarrow \mathfrak{R}_{\text{ev}}$ by

$$-\gamma(t) = \gamma(a + b - t)$$

Then if \mathfrak{U}_{ev} is open in \mathfrak{R}_{ev} with $\gamma([a, b]) \subset \mathfrak{U}_{\text{ev}}$ and $u : \mathfrak{U}_{\text{ev}} \rightarrow \mathfrak{R}$ is supersmooth,

$$\int_{-\gamma} dx u(x) = - \int_{\gamma} dx u(x).$$

Proof. Applying CVF on \mathbb{R} for $t = \phi(s)$ and $dt = \phi'(s)ds$, we have

$$\begin{aligned} \int_{\gamma \circ \phi} dx u(x) &= \int_c^d ds (\gamma(\phi(s)))' u(\gamma(\phi(s))) = \int_c^d ds \phi'(s) [\gamma'(\phi(s)) u(\gamma(\phi(s)))] \\ &= \int_a^b dt \gamma'(t) u(\gamma(t)) = \int_{\gamma} dx u(x). \end{aligned}$$

Other statements are also proved analogously. \square

Corollary 3.4 (Corollary 3.7 in [19] on \mathfrak{B}_L). *Let $u(x)$ be a supersmooth function defined on a even superdomain $\mathfrak{U}_{\text{ev}} \subset \mathfrak{R}^{1|0}$ into \mathfrak{C} .*

(a) *Let γ_1, γ_2 be continuous and piecewise C^1 -curves from $[a, b] \rightarrow \mathfrak{U}_{\text{ev}}$ such that $\lambda = \gamma_1(a) = \gamma_2(a)$ and $\mu = \gamma_1(b) = \gamma_2(b)$. If γ_1 is homotopic to γ_2 , then*

$$(3.3) \quad \int_{\gamma_1} dx u(x) = \int_{\gamma_2} dx u(x).$$

(b) *If $u : \mathfrak{R}_{\text{ev}} \rightarrow \mathfrak{R}$ is \mathcal{C}_{SS} on all \mathfrak{R}_{ev} , one denote it “unambiguously” as*

$$\int_{\lambda}^{\mu} dx u(x) = \int_{\gamma} dx u(x).$$

Here, $\gamma : [a, b] \rightarrow \mathfrak{R}_{\text{ev}}$ is any path in \mathfrak{R}_{ev} with $\gamma(a) = \lambda$, $\gamma(b) = \mu$.

Proposition 3.5. *For a given change of variable $x = \varphi(y)$, we define the pull-back of 1-form $\mathbf{v}_x = dx \rho(x)$ by $(\varphi^* \mathbf{v})_y = dy \frac{\partial \varphi(y)}{\partial y} \rho(\varphi(y))$. Then, for paths $\gamma : [a, b] \rightarrow \mathfrak{R}_x^{1|0}$, $\varphi^{-1} \circ \gamma : [a, b] \rightarrow \mathfrak{R}_y^{1|0}$ and u , we have*

$$\int_{\gamma} \mathbf{v} = \int_{\gamma} dx \mathbf{v}_x \rho(x) u(x) = \int_{\varphi^{-1} \circ \gamma} dy (\varphi^* \mathbf{v})_y \rho(\varphi(y)) u(\varphi(y)) = \int_{\varphi^{-1} \circ \gamma} \varphi^* \mathbf{v} \varphi^* u.$$

Proof. From

$$(3.4) \quad \begin{array}{ccc} [a, b] & \xrightarrow{\gamma} & \mathfrak{R}_x^{1|0} \\ \parallel & & \uparrow \varphi \\ [a, b] & \xrightarrow{\delta} & \mathfrak{R}_y^{1|0} \end{array} \quad \text{with} \quad \delta = \psi \circ \gamma \quad \text{and} \quad \psi = \varphi^{-1},$$

we have not only

$$\int_{\gamma} \mathbf{v}_x u(x) = \int_a^b dt \dot{\gamma}(t) \rho(\gamma(t)) u(\gamma(t)),$$

but also

$$\begin{aligned} \int_{\varphi^{-1} \circ \gamma} (\varphi^* \mathbf{v})_y \varphi^* u(y) &= \int_{\varphi^{-1} \circ \gamma} dy \frac{d\varphi(y)}{dy} \rho(\varphi(y)) u(\varphi(y)) \\ &= \int_a^b dt \frac{d}{dt} (\varphi^{-1}(\gamma(t))) \frac{d\varphi(y)}{dy} \rho(\varphi(y)) u(\varphi(y)) \Big|_{y=\varphi^{-1} \circ \gamma(t)} = \int_a^b dt \gamma'(t) \rho(\gamma(t)) u(\gamma(t)). \end{aligned}$$

Here, we used $y = \varphi^{-1}(\varphi(y)) = \psi(\varphi(y))$, $x = \gamma(t)$, $y = \psi(\gamma(t))$ with

$$1 = \frac{d\varphi(y)}{dy} \cdot \frac{d\psi(x)}{dx} \Big|_{x=\varphi(y)}, \quad \frac{d}{dt} (\psi(\gamma(t))) = \gamma'(t) \frac{d\psi(x)}{dx} \Big|_{x=\gamma(t)}, \quad \frac{d\psi(x)}{dx} \Big|_{x=\varphi(y)} = \left(\frac{d\varphi(y)}{dy} \right)^{-1}. \quad \square$$

Example 3.1 (Translational invariance). *Let $I = (a, b) \subset \mathbb{R}$. We put $\mathfrak{M} = \gamma(I) = \{x \in \mathfrak{R}_{\text{ev}} \mid \pi_{\text{B}}(x) = q \in I\} \subset \mathfrak{R}_{\text{ev}}$ by identifying $q \in I$ as $\gamma(q) = x \in \mathfrak{R}_{\text{ev}}$. Taking a non-zero nilpotent element $\nu \in \mathfrak{R}_{\text{ev}}$, that is, $0 \neq \nu$ and $\pi_{\text{B}}(\nu) = 0$, we put $\tau_{\nu} : \mathfrak{R}_{\text{ev}} \ni y \rightarrow x = \varphi(y) = \tau_{\nu}(y) = y - \nu \in \mathfrak{R}_{\text{ev}}$,*

$$\mathfrak{M}_1 = \tau_{\nu}^{-1}(\mathfrak{M}) = \{x + \nu \mathfrak{R}_{\text{ev}} \mid \pi_{\text{B}}(x) = q \in I\}, \quad \gamma_1(q) = \tau_{\nu}^{-1}(\gamma(q)).$$

Then, we have

$$\int_{\mathfrak{M}} dx u(x) = \int_a^b dq \gamma'(q) u(\gamma(q)) = \int_a^b dq \gamma_1'(q) u(\gamma(q)) = \int_{\mathfrak{M}_1} dy u(y - \nu).$$

Remark 3.6. (i) Above identification $\gamma(q) = x \in \mathfrak{R}_{\text{ev}}$ is obtained as the Grassmann continuation $\tilde{\iota}$ of a function $\iota(q) = q \in C^\infty(I : \mathbb{R})$. In fact,

$$\tilde{\iota}(x) = \sum_{\alpha} \frac{\partial^{\alpha} \iota(q)}{\partial q^{\alpha}} (x_B) x_S^{\alpha} = x_B + x_S = x.$$

(ii) As is noted in Example 2.2 of [19], there occurs an inconsistency when we apply the naive definition of integration (1.1): Let $\mathfrak{U}_{\text{ev}} = \pi_B^{-1}(a, b)$. Then we have

$$\int_{\mathfrak{U}_{\text{ev}}} dx x = \int_a^b dq q = \frac{1}{2}(b^2 - a^2).$$

For $0 \neq \nu \in \mathfrak{R}_{\text{ev}}$ with $\pi_B(\nu) = 0$, we have $\{x = y - \nu \mid y \in \pi_B^{-1}(a, b)\} = \mathfrak{U}_{\text{ev}}$, “ $dx = dy$ ” and therefore

$$\int_{\mathfrak{U}_{\text{ev}}} dy (y - \nu) = \int_a^b dq (q - \nu) = \frac{1}{2}(b^2 - a^2) - \nu(b - a) \neq \int_{\mathfrak{U}_{\text{ev}}} dx x.$$

This inconsistency stems from the naive definition (1.1), that is, Berezin’s integral w.r.t. even variables is an integral over \mathbb{R}^m and not on $\mathfrak{R}_{\text{ev}}^m$. By the nilpotency of ν , we have

$$(\nu + \tilde{b})^2 - (\nu + \tilde{a})^2 = \tilde{b}^2 - \tilde{a}^2 + 2\nu(\tilde{b} - \tilde{a}) = b^2 - a^2 + 2\nu(b - a),$$

which remedies this inconsistency by formally putting

$$\int_{\mathfrak{M}_1} dy u(\varphi(y)) = \int_{\tilde{a}+\nu}^{\tilde{b}+\nu} dy u(\varphi(y)) = \frac{1}{2}(b^2 - a^2).$$

Or more rigorous description is given in Example 3.1 above.

3.2. Many dimensional case.

3.2.1. *à la Rogers* [19]. We replace \mathfrak{B}_L or \mathfrak{B} with \mathfrak{R} in her arguments.

Definition 3.2 (*m-path*). Let $\mathbf{I}^m = \prod_{j=1}^m [a_j, b_j] \subset \mathbb{R}^m$. For a continuous and piecewise C^1 function $\gamma(t) : \mathbf{I}^m \rightarrow \mathfrak{R}_{\text{ev}}^m$ (called *m-path*) with $\gamma(t) = (\gamma_1(t), \dots, \gamma_m(t))$, $t = (t_1, \dots, t_m)$, we define

$$\int_{\gamma} dx_1 \cdots dx_m u(x) = \int_{\mathbf{I}^m} dt_1 \cdots dt_m \det(J(\gamma)(t)) u(\gamma(t)).$$

Here, $u : \mathfrak{U}_{\text{ev}} \rightarrow \mathfrak{R}$ is continuous on an open set $\mathfrak{U}_{\text{ev}} \subset \mathfrak{R}_{\text{ev}}^m$ containing $\gamma(\mathbf{I}^m)$ and

$$J(\gamma)(t) = \left(\frac{\partial \gamma_i(t)}{\partial t_j} \right)_{i,j=1, \dots, m}.$$

Proposition 3.7 (see, Theorem 4.4 of Rogers [19] on \mathfrak{B}_L). Let \mathfrak{U}_{ev} be open in $\mathfrak{R}_{\text{ev}}^m$ and let $\phi : \mathfrak{U}_{\text{ev}} \rightarrow \mathfrak{R}_{\text{ev}}^m$ be an injective and supersmooth mapping. And let $\gamma : \mathbf{I}^m \rightarrow \mathfrak{U}_{\text{ev}}$ be an *m-path* in \mathfrak{U}_{ev} and $u : \mathfrak{U}_{\text{ev}} \rightarrow \mathfrak{R}$ be continuous. Then

$$\int_{\gamma} dx u = \int_{\phi \circ \gamma} dx (\det J(\phi))^{-1} \cdot u \circ \phi^{-1} \quad \text{with} \quad J(\phi) = \frac{\partial \phi_i}{\partial x_j}$$

with $\phi(x) = (\phi_1(x), \dots, \phi_m(x))$, $x = (x_1, \dots, x_m)$.

3.2.2. *à la Vladimirov and Volovich* [25]. Since they use \mathfrak{B}_L in their arguments, we need some modifications to work on \mathfrak{R} .

Definition 3.3 (A *m-dimensional singular manifold and integrals on it*). Let \mathfrak{M} be *m-dimensional singular manifold*, that is, there exists a pair (M, γ) such that $\mathfrak{M} = \gamma(M)$ where M is an oriented region in \mathbb{R}^m and $\gamma : M \rightarrow \gamma(M) \subset \mathfrak{R}_{\text{ev}}^m$. For a given $\mathbf{v} = dx_1 \wedge \cdots \wedge dx_m \rho(x)$ with $\rho \in C_{\text{SS}}(\mathfrak{M} : \mathfrak{R})$ and $u \in C_{\text{SS}}(\mathfrak{M} : \mathfrak{R})$, if the integral of the right-hand side exists, we define

$$(3.5) \quad \int_{\mathfrak{M}} \mathbf{v} u = \int_{\mathfrak{M}} \mathbf{v}_x u(x) = \int_M \gamma^* \mathbf{v} \cdot \gamma^* u = \int_M dq \det \frac{\partial \gamma(q)}{\partial q} \rho(\gamma(q)) \cdot u(\gamma(q)),$$

and if $u = 1$, then the form \mathbf{v} is said to be integrable over the singular manifold $\mathfrak{M} = \gamma(M)$.

Definition 3.4. The pairs (M, γ) and (N, δ) are said to be equivalent if $\mathfrak{M} = \gamma(M) = \delta(N)$ and there exists a diffeomorphism $\phi : N \rightarrow M$ such that $\delta = \gamma \circ \phi$. Thus

$$(3.6) \quad \begin{array}{ccc} M & \xrightarrow{\gamma} & \mathfrak{M} \\ \phi \uparrow & & \parallel \\ N & \xrightarrow{\delta} & \mathfrak{N} \end{array} \implies \int_M \gamma^* \mathbf{v} = \int_N \delta^* \mathbf{v}.$$

This implies that not only the integral (3.5) doesn't depend on the choice of the pair (M, γ) in an equivalent class but also we may interpret the formula (3.6) as a change of variables formula as follows:

Let φ be a mapping of class $C^1(\mathfrak{D})$ of the neighborhood \mathfrak{D} of \mathfrak{N} in $\mathfrak{R}_{\text{ev}}^m$ and $\varphi^* \mathbf{v}$ is the pull-back of the superform \mathbf{v} under the mapping φ . Then

$$(3.7) \quad \begin{array}{ccc} M & \xrightarrow{\gamma} & (\mathfrak{M}, \mathbf{v}) \\ \phi \uparrow & & \uparrow \varphi \\ N & \xrightarrow{\delta} & (\mathfrak{N}, \varphi^* \mathbf{v}) \end{array} \implies \begin{aligned} \int_{\mathfrak{M}} \mathbf{v} &= \int_M dq \det J(\gamma)(q) \cdot (\gamma^* \mathbf{v})_1 \\ &= \int_N dq' \det J(\gamma \circ \phi)(q') \cdot (\gamma \circ \phi)^* \mathbf{v} = \int_{\mathfrak{N}} \varphi^* \mathbf{v}. \end{aligned}$$

That is, we have $\phi^{-1*} \delta^* \varphi^* = \gamma^*$.

Restricting above argument to the case when $\gamma = \delta = \tilde{\iota}$ and $\varphi = \text{Id}$, the equality (3.7) reduces to the ordinary change of variables formula:

$$\int_M dq \rho(q) u(q) = \int_N dq' \det J(\phi)(q') \cdot \rho(\phi(q')) u(\phi(q')) \quad \text{with} \quad \mathbf{v}_x = dx \rho(x).$$

Example 3.2 (One-dimensional singular manifold). Let $M = (0, 1)$. The integral of the superform

$$\mathbf{v} = \sum_{i=1}^m dx_i \rho_i(x)$$

of degree 1 along the curve $\mathfrak{M} = \{x = \gamma(q) \mid q \in M\} \subset \mathfrak{R}_{\text{ev}}^m$ where $\gamma \in C^1((0, 1) : \mathfrak{R}_{\text{ev}}^m)$, is determined by

$$\int_{\mathfrak{M}} \mathbf{v} = \sum_{i=1}^m \int_M dq \varphi'(q) \rho_i(\varphi(q)) = \int_M \varphi^* \mathbf{v}.$$

A composition of change of variables. Let \mathfrak{V} be a domain in $\mathfrak{R}_{\text{ev}}^m$ and $\varphi^{(1)} : \mathfrak{V} \rightarrow \tilde{\mathfrak{U}}$ be a diffeomorphism. Moreover, \mathfrak{U} be a domain in $\mathfrak{R}_{\text{ev}}^m$ and $\varphi^{(2)} : \tilde{\mathfrak{U}} \rightarrow \mathfrak{U}$ be a diffeomorphism.

$$\begin{array}{ccc} U & \xrightarrow{\gamma} & (\mathfrak{U}, \mathbf{v}), \\ \phi^{(2)} \uparrow & & \uparrow \varphi^{(2)} \\ \tilde{U} & \xrightarrow{\tilde{\gamma}} & (\tilde{\mathfrak{U}}, \tilde{\mathbf{v}}), \quad \text{with} \\ \phi^{(1)} \uparrow & & \uparrow \varphi^{(1)} \\ V & \xrightarrow{\delta} & (\mathfrak{V}, \mathbf{w}), \end{array} \quad \begin{aligned} \mathbf{v}_x &= dx \rho(x), \\ \tilde{\mathbf{v}}_{\tilde{y}} &= (\varphi^{(2)*} \mathbf{v})_{\tilde{y}} = d\tilde{y} \det J(\varphi^{(2)})(\tilde{y}) \cdot \rho(\varphi^{(2)}(\tilde{y})), \\ \mathbf{w}_y &= dy (\varphi^{(1)*} \varphi^{(2)*} \mathbf{v})_{\tilde{y}} = dy \det J(\varphi)(y) \cdot \rho(\varphi(y)), \\ &\text{where } \varphi = \varphi^{(2)} \circ \varphi^{(1)}. \end{aligned}$$

Then we have,

$$\int_{\tilde{\mathfrak{U}}} d\tilde{y} \rho(\tilde{y}) v(\tilde{y}) = \int_{\mathfrak{V}} dy \det J(\varphi^{(1)})(y) \cdot v(\varphi^{(1)}(y)) \quad \text{for } v \in \mathcal{C}_{\text{SS}}(\tilde{\mathfrak{U}} : \mathfrak{R})$$

and

$$\int_{\mathfrak{U}} dx u(x) = \int_{\tilde{\mathfrak{U}}} d\tilde{y} \det J(\varphi^{(2)})(\tilde{y}) \cdot u(\varphi^{(2)}(\tilde{y})) \quad \text{for } u \in \mathcal{C}_{\text{SS}}(\mathfrak{U} : \mathfrak{R}).$$

Since $\det J(\varphi^{(1)})(y) \cdot \det J(\varphi^{(2)})(\tilde{y})|_{\tilde{y}=\varphi^{(1)}(y)} = \det J(\varphi^{(2)} \circ \varphi^{(1)})(y)$, we have

$$\int_{\mathfrak{U}} dx u(x) = \int_{\mathfrak{Y}} dy \det J(\varphi^{(2)} \circ \varphi^{(1)})(y) \cdot u(\varphi^{(2)} \circ \varphi^{(1)}(y)).$$

Therefore, we have

$$\int_{\varphi(\mathfrak{Y})} dx u(x) = \int_{\mathfrak{Y}} dy \det J(\varphi)(y) \cdot u(\varphi(y)).$$

4. INTEGRATION W.R.T. ODD VARIABLES

It seems natural to put formally

$$d\theta_j = \sum_{\mathbf{I} \in \mathbf{I}, |\mathbf{I}|=\text{od}} d\theta_{j,\mathbf{I}} \sigma^{\mathbf{I}} \quad \text{for} \quad \theta_j = \sum_{\mathbf{J} \in \mathbf{J}, |\mathbf{J}|=\text{od}} \theta_{j,\mathbf{J}} \sigma^{\mathbf{J}}.$$

Remark 4.1. Since above sum $\sum_{\mathbf{I}}$ stands for the position in the sequence space ω of Köthe and the element of it is given by $d\theta_{j,\mathbf{J}}$ for $|\mathbf{J}|$ is finite, we may give the meaning to $d\theta_j$.

Then, we have

$$d\theta_j \wedge d\theta_k = d\theta_k \wedge d\theta_j.$$

This makes us imagine that even if there exists the notion of integration, it differs much from the standard one on \mathbb{R}^m .

4.1. Berezin integral. We follow Vladimirov and Volovich [25], modifying it if necessary. Since the supersmooth functions on $\mathfrak{R}^{0|n}$ are characterized as the polynomials with value in \mathfrak{C} , we need to define the integrability for those under the conditions that

- (i) integrability of all polynomials,
- (ii) linearity of an integral, and
- (iii) invariance of the integral w.r.t. shifts.

Put $\mathcal{P}_n = \mathcal{P}_n(\mathfrak{C}) = \{u(\theta) = \sum_{a \in \{0,1\}^n} \theta^a u_a \mid u_a \in \mathfrak{C}\}$.

We say a mapping $I_n : \mathcal{P}_n \rightarrow \mathfrak{C}$ is an integral if it satisfies

- (1) \mathfrak{C} -linearity (from the right): $I_n(u\alpha + v\beta) = I_n(u)\alpha + I_n(v)\beta$ for $\alpha, \beta \in \mathfrak{C}$, $u, v \in \mathcal{P}_n$.
- (2) translational invariance: $I_n(u(\cdot + \omega)) = I_n(u)$ for all $\omega \in \mathfrak{R}^{0|n}$ and $u \in \mathcal{P}_n$.

Theorem 4.2. For the existence of the integral I_n satisfying above conditions (1) and (2), it is necessary and sufficient that

$$(4.1) \quad I_n(\phi_a) = 0 \quad \text{for} \quad \phi_a(\theta) = \theta^a, \quad |a| \leq n-1.$$

Moreover, we have

$$I_n(u) = \frac{\partial}{\partial \theta_n} \cdots \frac{\partial}{\partial \theta_1} u(\theta) \Big|_{\theta=0} I_n(\phi_{\bar{1}}) \quad \text{where} \quad \phi_{\bar{1}}(\theta) = \theta^{\bar{1}} = \theta_1 \cdots \theta_n.$$

Proof. If there exists I_n satisfying (1) and (2), then we have

$$I_n(v) = \sum_{|a| \leq n} I_n(\phi_a) v_a \quad \text{for} \quad v(\theta) = \sum_{|a| \leq n} \theta^a v_a = \sum_{|a| \leq n} \phi_a(\theta) v_a.$$

As

$$(\theta + \omega)^a = \theta^a + \sum_{|a-b| \geq 1, b \leq a} (-1)^* \theta^b \omega^{a-b},$$

$$I_n(v(\cdot + \omega)) = \sum_{|a| \leq n} I_n(\phi_a(\cdot + \omega))v_a = \sum_{|a| \leq n} I_n(\phi_a)v_a + \sum_{|a| \leq n} \sum_{|a-b| \geq 1, b \leq a} (-1)^* I_n(\phi_b)v_b \omega^{a-b},$$

by virtue of (2), we have

$$\sum_{|a| \leq n} \sum_{|a-b| \geq 1, b \leq a} (-1)^* I_n(\phi_b)v_b \omega^{a-b} = 0.$$

Here, $v_a \in \mathfrak{C}$ and $\omega \in \mathfrak{R}_{\text{od}}^n$ are arbitrary, we have (4.1). Converse is obvious. \square

Definition 4.1. We put $I_n(\phi_{\bar{1}}) = 1$, i.e.,

$$\int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1 \theta_1 \cdots \theta_n = 1.$$

Therefore, we put, for any $v = \sum_{|a| \leq n} \theta^a v_a \in \mathcal{P}_n(\mathfrak{C})$

$$I_n(v) = \int_{\mathfrak{R}^{0|n}} d\theta v(\theta) = \int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1 v(\theta_1, \dots, \theta_n) = (\partial_{\theta_n} \cdots \partial_{\theta_1} v)(0) = v_{\bar{1}} = \int_{\text{Berezin}} d^n \theta f(\theta).$$

This is called the (Berezin) integral of v on $\mathfrak{R}^{0|n}$.

Then, we have

Proposition 4.3. Given $v, w \in \mathcal{P}_n(\mathfrak{C})$, we have the following:

(1) (\mathfrak{C} -linearity) For any homogeneous $\lambda, \mu \in \mathfrak{C}$,

$$(4.2) \quad \int_{\mathfrak{R}^{0|n}} d\theta (\lambda v + \mu w)(\theta) = (-1)^{np(\lambda)} \lambda \int_{\mathfrak{R}^{0|n}} d\theta v(\theta) + (-1)^{np(\mu)} \mu \int_{\mathfrak{R}^{0|n}} d\theta w(\theta).$$

(2) (Translational invariance) For any $\rho \in \mathfrak{R}^{0|n}$, we have

$$(4.3) \quad \int_{\mathfrak{R}^{0|n}} d\theta v(\theta + \rho) = \int_{\mathfrak{R}^{0|n}} d\theta v(\theta).$$

(3) (Integration by parts) For $v \in \mathcal{P}_n(\mathfrak{C})$ such that $p(v) = 1$ or 0, we have

$$(4.4) \quad \int_{\mathfrak{R}^{0|n}} d\theta v(\theta) \partial_{\theta_s} w(\theta) = -(-1)^{p(v)} \int_{\mathfrak{R}^{0|n}} d\theta (\partial_{\theta_s} v(\theta)) w(\theta).$$

(4) (Linear change of variables) Let $A = (A_{jk})$ with $A_{jk} \in \mathfrak{R}_{\text{ev}}$ be an invertible matrix. Then,

$$(4.5) \quad \int_{\mathfrak{R}^{0|n}} d\theta v(\theta) = (\det A)^{-1} \int_{\mathfrak{R}^{0|n}} d\omega v(A \cdot \omega).$$

(5) (Iteration of integrals)

$$(4.6) \quad \int_{\mathfrak{R}^{0|n}} d\theta v(\theta) = \int_{\mathfrak{R}^{0|n-k}} d\theta_n \cdots d\theta_{k+1} \left(\int_{\mathfrak{R}^{0|k}} d\theta_k \cdots d\theta_1 v(\theta_1, \dots, \theta_k, \theta_{k+1}, \dots, \theta_n) \right).$$

(6) (Odd change of variables) Let $\theta = \theta(\omega)$ be an odd change of variables such that $\theta(0) = 0$ and $\det \frac{\partial \theta(\omega)}{\partial \omega} \Big|_{\omega=0} \neq 0$. Then, for any $v \in \mathcal{P}_n(\mathfrak{C})$,

$$(4.7) \quad \int_{\mathfrak{R}^{0|n}} d\theta v(\theta) = \int_{\mathfrak{R}^{0|n}} d\omega \left(\det \frac{\partial \theta(\omega)}{\partial \omega} \right)^{-1} v(\theta(\omega)).$$

(7) (δ -function) For $v \in \mathcal{P}_n(\mathfrak{C})$ and $\omega \in \mathfrak{R}^{0|n}$,

$$(4.8) \quad \int_{\mathfrak{R}^{0|n}} d\theta (\omega_1 - \theta_1) \cdots (\omega_n - \theta_n) v(\theta) = v(\omega).$$

(4.8) allows us to put

$$\delta(\theta - \omega) = (\theta_1 - \omega_1) \cdots (\theta_n - \omega_n),$$

though $\delta(-\theta) = (-1)^n \delta(\theta)$.

We omit the proof, since we may apply the arguments in pp.755-757 of Vladimirov and Volovich [25] with slight modifications if necessary.

Remark 4.4. (i) We get the integration by parts formula, without the fundamental theorem of elementary analysis.

(ii) Moreover, since in conventional integration we get $\int dy f(y) = a \int dx f(ax)$, therefore the formula in (4.5) is very different from usual one. Analogous difference appears in (4.7).

5. INTEGRATION W.R.T. EVEN AND ODD VARIABLES

5.1. Proof of Theorem 1.3. For future use, we give a precise proof of Berezin [1] and Rogers [23] because their proofs are not so easy to understand at least for a tiny little old mathematician.

First of all, we prepare

Lemma 5.1. Let $u(x, \theta) = \sum_{|a| \leq n} \theta^a u_a(x)$ be supersmooth on $\mathfrak{U} = \mathfrak{U}_{\text{ev}} \times \mathfrak{R}_{\text{od}}^n$. If $\int_{\mathfrak{U}_{\text{ev}}} dx u_a(x)$ exists for each a , then we have

$$\mathfrak{B}\text{-}\iint_{\mathfrak{U}} dx d\theta u(x, \theta) = \int_{\mathfrak{U}_{\text{ev}}} dx \left[\int_{\mathfrak{R}_{\text{od}}^n} d\theta u(x, \theta) \right] = \int_{\mathfrak{R}_{\text{od}}^n} d\theta \left[\int_{\mathfrak{U}_{\text{ev}}} dx u(x, \theta) \right].$$

Proof. By the primitive definition of integral, we have

$$\mathfrak{B}\text{-}\iint_{\mathfrak{U}} dx d\theta u(x, \theta) = \int_{\mathfrak{U}_{\text{ev}}} dx \left[\int_{\mathfrak{R}_{\text{od}}^n} d\theta u(x, \theta) \right] = \int_{\mathfrak{U}_{\text{ev}}} dx u_{\bar{1}}(x),$$

and

$$\int_{\mathfrak{R}_{\text{od}}^n} d\theta \sum_{|a| \leq n} \left[\int_{\mathfrak{U}_{\text{ev}}} dx \theta^a u_a(x) \right] = \int_{\mathfrak{R}_{\text{od}}^n} d\theta \sum_{|a| \leq n} \theta^a \left[\int_{\mathfrak{U}_{\text{ev}}} dx u_a(x) \right] = \int_{\mathfrak{U}_{\text{ev}}} dx u_{\bar{1}}(x). \quad \square$$

(I) Now, we consider a simple case: Let a linear coordinate change be given by

$$(x, \theta) = (y, \omega)M, \quad M = \begin{pmatrix} A & C \\ D & B \end{pmatrix}.$$

That is,

$$x_i = \sum_{k=1}^m y_k A_{ki} + \sum_{\ell=1}^n \omega_\ell D_{\ell i} = x_i(y, \omega), \quad \theta_j = \sum_{k=1}^m y_k C_{kj} + \sum_{\ell=1}^n \omega_\ell B_{\ell j} = \theta_j(y, \omega)$$

with $A_{ki}, B_{\ell j} \in \mathfrak{C}_{\text{ev}}$ and $C_{\ell i}, D_{kj} \in \mathfrak{C}_{\text{od}}$, and we have

$$(5.1) \quad \text{sdet} \left(\frac{\partial(x, \theta)}{\partial(y, \omega)} \right) = \det A \cdot \det^{-1}(B - DA^{-1}C) = \det(A - CB^{-1}D) \cdot \det^{-1}B = \text{sdet} M.$$

Interchanging the order of integration, putting $\omega^{(1)} = \omega B$ and $y^{(1)} = yA$, we get

$$\begin{aligned} \mathfrak{B}\text{-}\iint dy d\omega u(yA + \omega D, yC + \omega B) &= \int dy \left[\int d\omega u(yA + \omega D, yC + \omega B) \right] \\ &= \int dy \left[\int d\omega^{(1)} \det B \cdot u(yA + \omega^{(1)} B^{-1} D, yC + \omega^{(1)}) \right] \\ &= \int d\omega^{(1)} \det B \left[\int dy u(yA + \omega^{(1)} B^{-1} D, yC + \omega^{(1)}) \right] \\ &= \int d\omega^{(1)} \det B \left[\int dy^{(1)} \det A^{-1} \cdot u(y^{(1)} + \omega^{(1)} B^{-1} D, y^{(1)} A^{-1} C + \omega^{(1)}) \right], \end{aligned}$$

that is, since

$$\frac{\partial(y, \omega)}{\partial(y^{(1)}, \omega^{(1)})} = \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}, \quad \text{sdet} \left(\frac{\partial(y, \omega)}{\partial(y^{(1)}, \omega^{(1)})} \right) = \det A^{-1} \cdot \det B,$$

we have

$$(5.2) \quad \begin{aligned} & \mathfrak{B}\!\!\!\!\!\int dy d\omega u(yA + \omega D, yC + \omega B) \\ &= \mathfrak{B}\!\!\!\!\!\int dy^{(1)} d\omega^{(1)} \text{sdet} \left(\frac{\partial(y, \omega)}{\partial(y^{(1)}, \omega^{(1)})} \right) \cdot u(y^{(1)} + \omega^{(1)} B^{-1} D, y^{(1)} A^{-1} C + \omega^{(1)}). \end{aligned}$$

Analogously, using Lemma 5.1 and by introducing change of variables as

$$y^{(2)} = y^{(1)}, \omega^{(2)} = \omega^{(1)} + y^{(1)} A^{-1} C \implies \text{sdet} \left(\frac{\partial(y^{(1)}, \omega^{(1)})}{\partial(y^{(2)}, \omega^{(2)})} \right) = \text{sdet} \begin{pmatrix} 1 & -A^{-1} C \\ 0 & 1 \end{pmatrix} = 1,$$

we get

$$(5.3) \quad \begin{aligned} & \mathfrak{B}\!\!\!\!\!\int dy^{(1)} d\omega^{(1)} u(y^{(1)} + \omega^{(1)} B^{-1} D, y^{(1)} A^{-1} C + \omega^{(1)}) \\ &= \mathfrak{B}\!\!\!\!\!\int dy^{(2)} d\omega^{(2)} \text{sdet} \left(\frac{\partial(y^{(1)}, \omega^{(1)})}{\partial(y^{(2)}, \omega^{(2)})} \right) \cdot u(y^{(2)} + (\omega - y^{(2)} A^{-1} C) B^{-1} D, \omega^{(2)}). \end{aligned}$$

Then by

$$\begin{aligned} & y^{(3)} = y^{(2)}(1 - A^{-1} C B^{-1} D), \omega^{(3)} = \omega^{(2)} \\ & \implies \text{sdet} \left(\frac{\partial(y^{(2)}, \omega^{(2)})}{\partial(y^{(3)}, \omega^{(3)})} \right) = \text{sdet} \begin{pmatrix} (1 - A^{-1} C B^{-1} D)^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \det^{-1}(1 - A^{-1} C B^{-1} D), \end{aligned}$$

we have

$$(5.4) \quad \begin{aligned} & \mathfrak{B}\!\!\!\!\!\int dy^{(2)} d\omega^{(2)} u(y^{(2)} + (\omega - y^{(2)} A^{-1} C) B^{-1} D, \omega^{(2)}) \\ &= \mathfrak{B}\!\!\!\!\!\int dy^{(3)} d\omega^{(3)} \text{sdet} \left(\frac{\partial(y^{(2)}, \omega^{(2)})}{\partial(y^{(3)}, \omega^{(3)})} \right) \cdot u(y^{(3)} + \omega^{(3)} B^{-1} D, \omega^{(3)}). \end{aligned}$$

Finally by

$$x = y^{(3)} + \omega^{(3)} B^{-1} D, \theta = \omega^{(3)} \implies \text{sdet} \left(\frac{\partial(y^{(3)}, \omega^{(3)})}{\partial(x, \theta)} \right) = \text{sdet} \begin{pmatrix} 1 & 0 \\ -B^{-1} D & 1 \end{pmatrix} = 1,$$

using $\det B \det^{-1}(A - C B^{-1} D) \cdot (\det A \det^{-1}(B - D A^{-1} C)) = 1$ from (5.1), we have,

$$(5.5) \quad \mathfrak{B}\!\!\!\!\!\int dy d\omega u(yA + \omega D, yC + \omega B) = \text{sdet} M^{-1} \cdot \mathfrak{B}\!\!\!\!\!\int dx d\theta \text{sdet} \left(\frac{\partial(y^{(3)}, \omega^{(3)})}{\partial(x, \theta)} \right) \cdot u(x, \theta). \quad //$$

Remark 5.2. For the linear change of variables, it is not necessary to assume the compactness of support for integrand using primitive definition of integration.

(II) (ii-a) If H_1 and H_2 are superdiffeomorphisms of open subsets of $\mathfrak{R}^{m|n}$ with the image of H_1 equals to the domain of H_2 , then

$$\text{Ber}(H_1) \cdot \text{Ber}(H_2) = \text{Ber}(H_2 \circ H_1) \quad \text{where} \quad \text{Ber}(H)(y, \omega) = \text{sdet} J(H)(y, \omega).$$

Here, for $H(y, \omega) = (x_k(y, \omega), \theta_l(y, \omega)) : \mathfrak{R}^{m|n} \rightarrow \mathfrak{R}^{m|n}$, we put

$$J(H)(y, \omega) = \begin{pmatrix} \frac{\partial x_k(y, \omega)}{\partial y_i} & \frac{\partial \theta_l(y, \omega)}{\partial y_i} \\ \frac{\partial x_k(y, \omega)}{\partial \omega_j} & \frac{\partial \theta_l(y, \omega)}{\partial \omega_j} \end{pmatrix} = \frac{\partial(x, \theta)}{\partial(y, \omega)}.$$

(ii-b) Any superdiffeomorphism of an open subset of $\mathfrak{R}^{m|n}$ may be decomposed as $H = H_2 \circ H_1$ where

$$(5.6) \quad \begin{cases} H_1(y, \omega) = (h_1(y, \omega), \omega) = (\tilde{y}, \tilde{\omega}) & \text{with } h_1 : \mathfrak{R}^{m|n} \rightarrow \mathfrak{R}^{m|0}, \\ H_2(\tilde{y}, \tilde{\omega}) = (\tilde{y}, h_2(\tilde{y}, \tilde{\omega})) & \text{with } h_2 : \mathfrak{R}^{m|n} \rightarrow \mathfrak{R}^{0|n}. \end{cases}$$

Remark 5.3. (i) If $H(y, \omega) = (h_1(y, \omega), h_2(y, \omega))$ is given by $h_1(y, \omega) = yA + \omega D$ and $h_2(y, \omega) = yC + \omega B$ as above, putting $H_1(y, \omega) = (yA + \omega D, \omega) = (\tilde{y}, \omega)$ and $H_2(\tilde{y}, \omega) = (\tilde{y}, \tilde{y}A^{-1}C + \omega(B - DA^{-1}C))$, we have $H = H_2 \circ H_1$. In this case, we rewrite the procedures (5.2)–(5.5) as

$$\begin{aligned} & \mathfrak{B}\text{-}\iint dyd\omega u(yA + \omega D, yC + \omega B) \\ &= \mathfrak{B}\text{-}\iint d\tilde{y}d\tilde{\omega} \text{sdet} \left(\frac{\partial(y, \omega)}{\partial(\tilde{y}, \omega)} \right) \cdot u(\tilde{y}, (\tilde{y} - \omega D)A^{-1}C + \omega B) \quad \text{with} \quad \tilde{y} = yA + \omega D \\ &= \det A^{-1} \cdot \mathfrak{B}\text{-}\iint dx d\theta \text{sdet} \left(\frac{\partial(\tilde{y}, \omega)}{\partial(x, \theta)} \right) \cdot u(x, \theta) \quad \text{with} \quad x = \tilde{y}, \theta = \tilde{y}A^{-1}C + \omega(B - DA^{-1}C) \\ &= \det A^{-1} \cdot \det(B - DA^{-1}C) \cdot \mathfrak{B}\text{-}\iint dx d\theta u(x, \theta). \end{aligned}$$

(ii) Analogously, putting $H_1(y, \omega) = (y, yC + \omega B) = (y, \theta)$ and $H_2(y, \theta) = (y(A - CB^{-1}D) + \theta B^{-1}D, \theta)$, we have $H = H_2 \circ H_1$, and

$$\begin{aligned} & \mathfrak{B}\text{-}\iint dyd\omega u(yA + \omega D, yC + \omega B) \\ &= \mathfrak{B}\text{-}\iint dyd\theta \text{sdet} \left(\frac{\partial(y, \omega)}{\partial(y, \theta)} \right) \cdot u(y(A - CB^{-1}D) + \theta B^{-1}D, \theta) \quad \text{with} \quad \theta = yC + \omega B \\ &= \det B \cdot \mathfrak{B}\text{-}\iint dx d\theta \text{sdet} \left(\frac{\partial(y, \theta)}{\partial(x, \theta)} \right) \cdot u(x, \theta) \quad \text{with} \quad x = y(A - CB^{-1}D) + CB^{-1}\theta \\ &= \det B \cdot \det^{-1}(A - CB^{-1}D) \cdot \mathfrak{B}\text{-}\iint dx d\theta u(x, \theta). \end{aligned}$$

(iii) For any given superdiffeomorphism $H(y, \omega) = (h_1(y, \omega), h_2(y, \omega))$, we put

$$H_1(y, \omega) = (h_1(y, \omega), \omega) = (\tilde{y}, \omega).$$

Moreover, using the inverse function $y = g(\tilde{y}, \omega)$ of $\tilde{y} = h_1(y, \omega)$, we put $\tilde{h}_2(\tilde{y}, \omega) = h_2(g(\tilde{y}, \omega), \omega)$ and $H_2(\tilde{y}, \omega) = (\tilde{y}, \tilde{h}_2(\tilde{y}, \omega))$. Since $h_2(y, \omega) = \tilde{h}_2(h_1(y, \omega), \omega)$, we have $H = H_2 \circ H_1$. We denote $h_1(y, \omega) = (h_{1j}(y, \omega)) = (h_{11}, \dots, h_{1m})$ and $h_2(y, \omega) = (h_{2\ell}(y, \omega)) = (h_{21}, \dots, h_{2n})$. Then, for $k, \ell = 1, \dots, n$,

$$\frac{\partial \tilde{h}_{2\ell}}{\partial \omega_k} = \frac{\partial h_{2\ell}}{\partial \omega_k} + \sum_{i=1}^m \frac{\partial g_i}{\partial \omega_k} \frac{\partial h_{2\ell}}{\partial y_i}$$

with

$$0 = \frac{\partial \tilde{y}_j}{\partial \omega_k} = \frac{\partial h_{1j}(g(y, \omega), \omega)}{\partial \omega_k} = \sum_{j=1}^m \frac{\partial g_i}{\partial \omega_k} \frac{\partial h_{1j}}{\partial y_i} + \frac{\partial h_{1j}}{\partial \omega_k},$$

we get

$$\frac{\partial \tilde{h}_{2\ell}}{\partial \omega_k} = \frac{\partial h_{2\ell}}{\partial \omega_k} - \sum_{i,j=1}^m \frac{\partial h_{1j}}{\partial \omega_k} \left(\frac{\partial h_{1j}}{\partial y_i} \right)^{-1} \frac{\partial h_{2\ell}}{\partial y_i}.$$

Therefore,

$$\text{Ber } H = \text{sdet} \left(\frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial \omega} \right) = \det \frac{\partial h_1}{\partial y} \cdot \det^{-1} \left(\frac{\partial h_2}{\partial \omega} - \frac{\partial h_1}{\partial \omega} \left(\frac{\partial h_1}{\partial y} \right)^{-1} \frac{\partial h_2}{\partial y} \right) = \det \frac{\partial h_1}{\partial y} \cdot \det^{-1} \frac{\partial \tilde{h}_2}{\partial \omega}.$$

(III) For each type of superdiffeomorphisms H_1 and H_2 , we need to prove the formula.

(III-1) Let $H(y, \omega) = (h(y, \omega), \omega)$ where $h = (h_j)_{j=1}^m : \mathfrak{R}^{m|n} \rightarrow \mathfrak{R}^{m|0}$. Then it is clear that

$$\text{Ber}(H)(y, \omega) = \det \left(\frac{\partial h_j(y, \omega)}{\partial y_i} \right) = \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) \prod_{i=1}^m \frac{\partial h_{\sigma(i)}(y, \omega)}{\partial y_i}.$$

For any $u(x, \theta) = \sum_{|a| \leq n} \theta^a u_a(x)$, we put

$$\mathfrak{B}\text{-}\iint_{\mathfrak{U}} dx d\theta u(x, \theta) = \int_{\mathfrak{U}_{\text{ev}, B}} dx \left(\int_{\mathfrak{R}^{0|n}} d\theta u(x, \theta) \right) = \int_{\mathfrak{U}_{\text{ev}, B}} dx u_{\bar{1}}(x).$$

On the other hand, we have

$$(5.7) \quad \begin{aligned} & \mathfrak{B} \text{--} \iint_{\mathfrak{Y}} dy d\omega \operatorname{Ber}(H)(y, \omega) (u \circ H)(y, \omega) \\ &= \int_{\pi_B(\mathfrak{Y})} dy \frac{\partial}{\partial \omega_n} \cdots \frac{\partial}{\partial \omega_1} \left(\det \left(\frac{\partial h_j(y, \omega)}{\partial y_i} \right) u(h(y, \omega), \omega) \right) \Big|_{\omega=0} = (I) + (II) \end{aligned}$$

with

$$\begin{aligned} (I) &= \int_{\pi_B(\mathfrak{Y})} dy \left(\det \left(\frac{\partial h_j(y, 0)}{\partial y_i} \right) u_{\bar{1}}(h(y, 0)) \right), \\ (II) &= \int_{\pi_B(\mathfrak{Y})} dy \frac{\partial}{\partial \omega_n} \cdots \frac{\partial}{\partial \omega_1} \left(\sum_{|a| < n} \omega^a u_a(h(y, \omega)) \det \left(\frac{\partial h_j(y, \omega)}{\partial y_i} \right) \right) \Big|_{\omega=0}. \end{aligned}$$

Applying the standard integration on \mathbb{R}^m to (I), we have readily

$$\int_{\pi_B(\mathfrak{Y})} dy \left(\det \left(\frac{\partial h_j(y, 0)}{\partial y_i} \right) u_{\bar{1}}(h(y, 0)) \right) = \int_{\mathfrak{U}_{\text{ev}, B}} dx u_{\bar{1}}(x) \quad \text{where } \mathfrak{U} = H(\mathfrak{Y}).$$

Claim 5.1. (II) of (5.7) equals to the total derivatives of even variables. More precisely, we have, for $u(x, \theta) = \sum_{|a| \leq n} \theta^a u_a(x)$,

$$\frac{\partial}{\partial \omega_n} \cdots \frac{\partial}{\partial \omega_1} \left(\sum_{|a| < n} \omega^a u_a(h(y, \omega)) \operatorname{Ber}(H)(y, \omega) \right) \Big|_{\omega=0} = \sum_{j=1}^m \frac{\partial}{\partial y_j} (*).$$

Remark 5.4. Though Rogers gives this claim in one sentence, line 3 from the bottom of p.142 of [23], we give a long and naive proof.

As $h_j(y, \omega) \in \mathfrak{R}_{\text{ev}}$, we have

$$\begin{aligned} h_j(y, \omega) &= h_{j\bar{0}}(y) + \sum_{|c|=\text{ev} \geq 2} \omega^c h_{j,c}(y), \\ u_a(h(y, \omega)) &= u_a(h_{\bar{0}}(y)) + \sum_{|c|=\text{ev} \geq 2} \omega^c h_{j,c}(y) u_{a,x_j}(h_{\bar{0}}(y)) + \sum_{|\alpha| \geq 2} \frac{\partial_x^\alpha u_a(h_{\bar{0}}(y))}{\alpha!} \left(\sum_{|c|=\text{ev} \geq 2} \omega^c h_{j,c}(y) \right)^\alpha, \\ \operatorname{Ber}(H)(y, \omega) &= \det \left(\frac{\partial h_j(y, \omega)}{\partial y_i} \right) = \sum_{\sigma \in \wp_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \frac{\partial h_{\sigma(i)}(y, \omega)}{\partial y_i} \\ &= \det \left(\frac{\partial h_{j,\bar{0}}(y)}{\partial y_i} \right) + \sum_{\sigma \in \wp_m} \operatorname{sgn}(\sigma) \sum_{j=1}^m \sum_{|c|=\text{ev} \geq 2} \omega^c \frac{\partial h_{\sigma(j),c}(y)}{\partial y_j} \prod_{i=1, i \neq j}^m \frac{\partial h_{\sigma(i),\bar{0}}(y)}{\partial y_i} \\ &\quad + \sum_{\sigma \in \wp_m} \operatorname{sgn}(\sigma) \sum_{j,k=1}^m \sum_{\substack{|c_j|=\text{ev} \\ |c_1+c_2|=|c| \geq 4}} \omega^c \frac{\partial h_{\sigma(j),c_1}(y)}{\partial y_j} \frac{\partial h_{\sigma(k),c_2}(y)}{\partial y_k} \prod_{i=1, i \neq j,k}^m \frac{\partial h_{\sigma(i),\bar{0}}(y)}{\partial y_i} + \text{etc.} \end{aligned}$$

Putting $\bar{1} - a = b$ or $= c_1 + c_2, = c_1 + c_2 + c_3$, etc, we have

$$(5.8) \quad \text{the coefficient of } \omega^b \text{ of } u_a(h(y, \omega)) \operatorname{Ber}(H)(y, \omega) = \text{I} + \text{II} + \text{III}$$

where

$$\begin{aligned} \text{I} &= \sum_{j=1}^m h_{j,b}(y) u_{a,x_j}(h_{\bar{0}}(y)) \sum_{\sigma \in \wp_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \frac{\partial h_{\sigma(i),\bar{0}}(y)}{\partial y_i}, \\ \text{II} &= u_a(h_{\bar{0}}(y)) \sum_{\sigma \in \wp_m} \operatorname{sgn}(\sigma) \sum_{j=1}^m \frac{\partial h_{\sigma(j),b}(y)}{\partial y_j} \prod_{i=1, i \neq j}^m \frac{\partial h_{\sigma(i),\bar{0}}(y)}{\partial y_i}, \\ \text{III} &= u_a(h_{\bar{0}}(y)) \sum_{\sigma \in \wp_m} \operatorname{sgn}(\sigma) \sum_{j,k=1}^m \sum_{b=c_1+c_2} \frac{\partial h_{\sigma(j),c_1}(y)}{\partial y_j} \frac{\partial h_{\sigma(k),c_2}(y)}{\partial y_k} \prod_{i=1, i \neq j,k}^m \frac{\partial h_{\sigma(i),\bar{0}}(y)}{\partial y_i} + \text{etc.} \end{aligned}$$

The term II is calculated as

$$\Pi = \sum_{j=1}^m \frac{\partial}{\partial y_j} \left[u_a(h_{\bar{0}}(y)) \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) h_{\sigma(j),b}(y) \prod_{i=1, i \neq j}^m \frac{\partial h_{\sigma(i),\bar{0}}(y)}{\partial y_i} \right] - A - B$$

where

$$\begin{aligned} A &= \sum_{j=1}^m \left(\sum_{k=1}^m \frac{\partial h_{k,\bar{0}}(y)}{\partial y_j} u_{a,x_k}(h_{\bar{0}}(y)) \right) \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) h_{\sigma(j),b}(y) \prod_{i=1, i \neq j}^m \frac{\partial h_{\sigma(i),\bar{0}}(y)}{\partial y_i}, \\ B &= \sum_{j=1}^m u_a(h_{\bar{0}}(y)) \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) h_{\sigma(j),b}(y) \frac{\partial}{\partial y_j} \left(\prod_{i=1, i \neq j}^m \frac{\partial h_{\sigma(i),\bar{0}}(y)}{\partial y_i} \right). \end{aligned}$$

Now, we want to prove (i) $A = I$, (ii) $B = 0$ and (iii) $III = 0$.

(i) To prove $A = I$, for each $k = 1, \dots, m$, we take all sums w.r.t. $\sigma \in \wp_m$ and j such that $\sigma(j) = k$. Then, relabeling in A , we have

$$\begin{aligned} \sum_{\sigma \in \wp_m} \sum_{j=1}^m \frac{\partial h_{\sigma(j),\bar{0}}(y)}{\partial y_j} u_{a,x_k}(h_{\bar{0}}(y)) \text{sgn}(\sigma) h_{k,b}(y) \prod_{i=1, i \neq j}^m \frac{\partial h_{\sigma(i),\bar{0}}(y)}{\partial y_i} \\ = u_{a,x_k}(h_{\bar{0}}(y)) h_{k,b}(y) \sum_{\sigma \in \wp_m} \text{sgn}(\sigma) \prod_{i=1}^m \frac{\partial h_{\sigma(i),\bar{0}}(y)}{\partial y_i}. \end{aligned}$$

(ii) Take two permutations σ and $\tilde{\sigma}$ in \wp_m such that

$$\sigma(i) = \tilde{\sigma}(j), \quad \sigma(j) = \tilde{\sigma}(i), \quad , \sigma(k) = \tilde{\sigma}(k) \quad \text{for } k \neq i, j, \quad \text{and} \quad \text{sgn}(\sigma) \text{sgn}(\tilde{\sigma}) = -1.$$

Then,

$$\text{sgn}(\sigma) h_{\sigma(j),b}(y) \frac{\partial}{\partial y_j} \left(\prod_{i=1, i \neq j}^m \frac{\partial h_{\sigma(i),\bar{0}}(y)}{\partial y_i} \right) + \text{sgn}(\tilde{\sigma}) h_{\tilde{\sigma}(j),b}(y) \frac{\partial}{\partial y_j} \left(\prod_{i=1, i \neq j}^m \frac{\partial h_{\tilde{\sigma}(i),\bar{0}}(y)}{\partial y_i} \right) = 0.$$

(iii) Interchanging the role of j, k and c_1, c_2 in III, we have $III = 0$. Others are treated analogously.

Therefore,

$$I + II + III = A + B = \sum_{j=1}^m \frac{\partial}{\partial y_j} (*)$$

and we have proved the claim above. //

Corollary 5.5. *If we assume the compactness of the support of $u_a(x)$ for $|a| < n$, then we get*

$$\int_{\pi_B(\mathfrak{U})} dy \frac{\partial}{\partial y_i} (u_a(h(y, \omega)) \partial_{\omega}^{\bar{1}-a} \text{Ber}(H)(y, \omega)) \Big|_{\omega=0} = 0.$$

(III-2) For $H(y, \omega) = (y, \phi(y, \omega))$ with $\phi(y, \omega) = (\phi_1(y, \omega), \dots, \phi_n(y, \omega)) \in \mathfrak{R}^{0|n}$, we may claim

$$(5.9) \quad \mathfrak{B} \! \! \! \int \! \! \! \int_{\mathfrak{Y}} dx d\theta u(x, \theta) = \mathfrak{B} \! \! \! \int \! \! \! \int_{\mathfrak{U}} dy d\omega \left(\det \left(\frac{\partial \phi_i}{\partial \omega_j} \right) \right)^{-1} u(y, \phi(y, \omega)).$$

In fact, by the analogous proof in (6) of Proposition 4.3, i.e. odd change of variables formula, we have the above readily. \square

5.2. Modification of Vladimirov and Volovich's approach. We need to check the well-definedness of Definition 1.3 given in the introduction. First of all, we remark that by the algebraic nature of integral

w.r.t. odd variables, we may interchange the order of integration as

$$\begin{aligned} \int_{\mathfrak{R}_{\text{od}}^n} d\vartheta \left[\int_{\Omega} dq \operatorname{sdet} J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)) \right] &= \frac{\partial}{\partial \vartheta_n} \cdots \frac{\partial}{\partial \vartheta_1} \int_{\Omega} dq \operatorname{sdet} J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)) \Big|_{\vartheta=0} \\ &= \int_{\Omega} dq \frac{\partial}{\partial \vartheta_n} \cdots \frac{\partial}{\partial \vartheta_1} (\operatorname{sdet} J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta))) \Big|_{\vartheta=0} \\ &= \int_{\Omega} dq \left[\int_{\mathfrak{R}_{\text{od}}^n} d\vartheta \operatorname{sdet} J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)) \right]. \end{aligned}$$

In case when $\gamma_{\bar{0}}(q, \vartheta)$ doesn't depend on ϑ , putting $\bar{\vartheta} = \gamma_{\bar{1}}(q, \vartheta)$ and $\bar{q} = \gamma_{\bar{0}}(q)$, we have

$$\begin{aligned} &\int_{\Omega} dq \frac{\partial}{\partial \vartheta_n} \cdots \frac{\partial}{\partial \vartheta_1} (\operatorname{sdet} J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta))) \Big|_{\vartheta=0} \\ &= \int_{\Omega} dq \det \left(\frac{\partial \gamma_{\bar{0}}(q)}{\partial q} \right) \left[\int_{\mathfrak{R}_{\text{od}}^n} d\vartheta \det^{-1} \left(\frac{\partial \gamma_{\bar{1}}(q, \vartheta)}{\partial \vartheta} \right) \cdot u(\gamma_{\bar{0}}(q), \gamma_{\bar{1}}(q, \vartheta)) \right] \\ &= \int_{\Omega} dq \det \left(\frac{\partial \gamma_{\bar{0}}(q)}{\partial q} \right) \left[\int_{\mathfrak{R}_{\text{od}}^n} d\bar{\vartheta} u(\gamma_{\bar{0}}(q), \bar{\vartheta}) \right] \\ &= \int d\bar{q} \int d\bar{\vartheta} u(\bar{q}, \bar{\vartheta}) = \int_{\gamma_{\bar{0}}(\Omega)} dx \left[\int_{\mathfrak{R}_{\text{od}}^n} d\theta u(x, \theta) \right]. \end{aligned}$$

That is, for $\mathfrak{M} = \gamma(\tilde{\Omega})$ with $\tilde{\Omega} = \Omega \times \mathfrak{R}_{\text{od}}^n$, $\gamma(q, \vartheta) = (\gamma_{\bar{0}}(q), \gamma_{\bar{1}}(q, \vartheta))$,

$$\begin{aligned} (5.10) \quad \mathfrak{W} \int \int_{\mathfrak{M}} dx d\theta u(x, \theta) &= \int_{\mathfrak{R}_{\text{od}}^n} d\theta \left[\int_{\gamma_{\bar{0}}(\Omega)} dx u(x, \theta) \right] = \int d\theta \left(\int dx u(x, \theta) \right) \\ &= \int_{\gamma_{\bar{0}}(\Omega)} dx \left[\int_{\mathfrak{R}_{\text{od}}^n} d\theta u(x, \theta) \right] = \int dx \left(\int d\theta u(x, \theta) \right). \end{aligned}$$

Moreover, we need to verify

Proposition 5.6 (Reparametrization invariance). *Let Ω and Ω' be domains in \mathbb{R}^m and we put $\tilde{\Omega}$ and $\tilde{\Omega}'$ as above. We assume $\tilde{\Omega}$ and $\tilde{\Omega}'$ are superdiffeomorphic each other, that is, there exist a diffeomorphism $\phi_{\bar{0}} : \Omega' \rightarrow \Omega$ such that $\frac{\partial \phi_{\bar{0}}(q')}{\partial q'}$ which is continuous in Ω' and $\det(\frac{\partial \phi_{\bar{0}}(q')}{\partial q'}) > 0$ and a map $\phi_{\bar{1}} : \Omega' \times \mathfrak{R}_{\text{od}}^n \ni (q', \vartheta') \rightarrow \phi_{\bar{1}}(q', \vartheta') \in \mathfrak{R}_{\text{od}}^n$ which is supersmooth w.r.t. ϑ' with $\det(\frac{\partial \phi_{\bar{1}}(q', \vartheta')}{\partial \vartheta'}) \neq 0$. Put*

$$\mathfrak{M}' = \{X' = (x', \theta') \mid X' = \gamma \circ \phi(q', \vartheta'), (q', \vartheta') \in \tilde{\Omega}'\} \quad \text{where} \quad \phi(q', \vartheta') = (\phi_{\bar{0}}(q'), \phi_{\bar{1}}(q', \vartheta')).$$

For a given path $\gamma : \tilde{\Omega} \rightarrow \mathfrak{R}^{m|n}$, we define a path $\gamma \circ \phi : \tilde{\Omega}' \rightarrow \mathfrak{R}^{m|n}$. Then, we have

$$\mathfrak{W} \int \int_{\gamma(\tilde{\Omega})} dx d\theta u(x, \theta) = \mathfrak{W} \int \int_{\gamma \circ \phi(\tilde{\Omega}')} dx' d\theta' u(x', \theta').$$

Proof. By definition, we have

$$\mathfrak{W} \int \int_{\gamma(\tilde{\Omega})} dx d\theta u(x, \theta) = \int_{\mathfrak{R}_{\text{od}}^n} d\vartheta \left(\int_{\Omega} dq \operatorname{sdet} J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)) \right)$$

and

$$\mathfrak{W} \int \int_{\gamma \circ \phi(\tilde{\Omega}')} dx' d\theta' u(x', \theta') = \int_{\mathfrak{R}_{\text{od}}^n} d\vartheta' \left(\int_{\Omega'} dq' \operatorname{sdet} J(\gamma \circ \phi)(q', \vartheta') \cdot u(\gamma \circ \phi(q', \vartheta')) \right).$$

Using

$$\gamma \circ \phi(q', \vartheta') = (\gamma_{\bar{0}}(\phi_{\bar{0}}(q')), \gamma_{\bar{1}}(q', \vartheta')), \gamma_{\bar{1}}(\phi_{\bar{0}}(q'), \phi_{\bar{1}}(q', \vartheta'))),$$

$$J(\gamma \circ \phi)(q', \vartheta') = J(\gamma)(\phi(q', \vartheta')) \cdot J(\phi)(q', \vartheta'),$$

$$\operatorname{sdet} J(\phi)(q', \vartheta') = \det^{-1} \left(\frac{\partial \phi_{\bar{1}}(q', \vartheta')}{\partial \vartheta'} \right) \cdot \det \left(\frac{\partial \phi_{\bar{0}}(q')}{\partial q'} \right),$$

we have

$$\operatorname{sdet} J(\gamma \circ \phi)(q', \vartheta') = \det \left(\frac{\partial \phi_{\bar{0}}(q')}{\partial q'} \right) \cdot \det^{-1} \left(\frac{\partial \phi_{\bar{1}}(q', \vartheta')}{\partial \vartheta'} \right) \cdot \operatorname{sdet} J(\gamma)(q, \vartheta) \Big|_{\substack{q=\phi_{\bar{0}}(q') \\ \vartheta=\phi_{\bar{1}}(q', \vartheta')}}.$$

Remarking the order of integration, we have

$$\begin{aligned}
& \int_{\mathfrak{R}_{\text{od}}^n} d\vartheta' \left(\int_{\Omega'} dq' \text{sdet } J(\gamma \circ \phi)(q', \vartheta') \cdot u(\gamma \circ \phi(q', \vartheta')) \right) \\
&= \int_{\Omega'} dq' \det \left(\frac{\partial \phi_{\bar{0}}(q')}{\partial q'} \right) \left[\int_{\mathfrak{R}_{\text{od}}^n} d\vartheta' \det^{-1} \left(\frac{\partial \phi_{\bar{1}}(q', \vartheta')}{\partial \vartheta'} \right) [\text{sdet } J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta))] \right] \Big|_{\substack{q=\phi_{\bar{0}}(q') \\ \vartheta=\phi_{\bar{1}}(q', \vartheta')}} \\
&= \int_{\Omega} dq \left[\int_{\mathfrak{R}_{\text{od}}^n} d\vartheta \text{sdet } J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)) \right] = \iint_{\tilde{\Omega}} dq d\vartheta \text{sdet } J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)). \quad \square
\end{aligned}$$

5.2.1. *Proof of Theorem 1.6 – change of variable formula under integral sign.*

$$\begin{array}{ccc}
(\tilde{\Omega}, dq d\vartheta) & \xrightarrow{\gamma} & (\mathfrak{M}, dx d\theta) \xrightarrow{u(x, \theta)} \mathfrak{W} \iint_{\mathfrak{M}} dx d\theta u(x, \theta) \in \mathfrak{R} \\
\parallel & \varphi \uparrow & \parallel \\
(\tilde{\Omega}, dq d\vartheta) & \xrightarrow{\delta} & (\mathfrak{N}, dy d\omega) \xrightarrow{\varphi^* u(y, \omega)} \mathfrak{W} \iint_{\varphi^{-1}(\mathfrak{M})} dy d\omega \text{sdet } J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)) \in \mathfrak{R}.
\end{array}$$

By definition, we have paths

$$\Omega \times \mathfrak{R}_{\text{od}}^n \ni (q, \vartheta) \rightarrow \gamma(q, \vartheta) = (x, \theta),$$

$$\Omega \times \mathfrak{R}_{\text{od}}^n \ni (q, \vartheta) \rightarrow \gamma_1(q, \vartheta) = (y, \omega),$$

which are related each other

$$(x, \theta) = \gamma(q, \vartheta) = \varphi(y, \omega) = \varphi(\gamma_1(q, \vartheta)), \quad \gamma_1 = \varphi^{-1} \circ \gamma.$$

Claim 5.2. *Denoting the pull-back of a “superform” as*

$$\mathfrak{v} = dx d\theta u(x, \theta) \rightarrow \varphi^* \mathfrak{v} = dy d\omega \text{sdet } J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)),$$

we have

$$(5.11) \quad \mathfrak{W} \iint_{\varphi^{-1} \circ \gamma(\tilde{\Omega})} \varphi^* \mathfrak{v} = \mathfrak{W} \iint_{\gamma(\tilde{\Omega})} \mathfrak{v}.$$

Proof. Since $J(\varphi^{-1} \circ \gamma) = J(\gamma) \cdot J(\varphi^{-1})$ which yields

$$\text{sdet } J(\varphi^{-1} \circ \gamma)(q, \vartheta) (\text{sdet } J(\varphi)(y, \omega)) \Big|_{(y, \omega) = \varphi^{-1} \circ \gamma(q, \vartheta)} = \text{sdet } J(\gamma)(q, \vartheta),$$

and by the definitions of path and integral, we have

$$\begin{aligned}
\mathfrak{W} \iint_{\varphi^{-1} \circ \gamma(\tilde{\Omega})} \varphi^* \mathfrak{v} &= \int_{\mathfrak{R}_{\text{od}}^n} d\vartheta \left[\int_{\Omega} dq \text{sdet } J(\varphi^{-1} \circ \gamma)(q, \vartheta) (\text{sdet } J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega))) \right. \\
&\quad \left. \Big|_{(y, \omega) = \varphi^{-1} \circ \gamma(q, \vartheta)} \right] \\
&= \int_{\mathfrak{R}_{\text{od}}^n} d\vartheta \left[\int_{\Omega} dq \text{sdet } J(\gamma)(q, \vartheta) \cdot u(\gamma(q, \vartheta)) \right] = \mathfrak{W} \iint_{\gamma(\tilde{\Omega})} \mathfrak{v}.
\end{aligned}$$

we have the claim. \square

Now, we interpret (5.11) as change of variables: Since we may denote integrals as

$$\mathfrak{W} \iint_{\gamma(\tilde{\Omega})} \mathfrak{v} = \mathfrak{W} \iint_{\mathfrak{M}} dx d\theta u(x, \theta),$$

and

$$\mathfrak{W} \iint_{\varphi^{-1} \circ \gamma(\tilde{\Omega})} \varphi^* \mathfrak{v} = \mathfrak{W} \iint_{\varphi^{-1} \mathfrak{M}} dy d\omega \text{sdet } J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)),$$

we have

$$\mathfrak{W} \iint_{\mathfrak{M}} dx d\theta u(x, \theta) = \mathfrak{W} \iint_{\varphi^{-1} \mathfrak{M}} dy d\omega \text{sdet } J(\varphi)(y, \omega) \cdot u(\varphi(y, \omega)). \quad \square$$

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